

Existence and asymptotic stability of quasi-periodic solutions of discrete NLS with potential

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Abstract

We prove the existence of a 2-parameter family of small quasi-periodic solutions of discrete nonlinear Schrödinger equation (DNLS). We further show that all small solutions of DNLS decouples to one of these quasi-periodic solutions and dispersive wave. As a byproduct, we show that all small nonlinear bound states including excited states are unstable.

1 Introduction

In this paper, we consider small solutions of the discrete nonlinear Schrödinger equation (DNLS):

$$i\partial_t u = Hu + |u|^6 u, \quad u : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{C}, \quad (1.1)$$

where, $H = -\Delta + V$, Δ is the discrete Laplacian:

$$(\Delta u)(n) := u(n+1) - 2u(n) + u(n-1),$$

and $\sum_{n \in \mathbb{Z}} (1 + |n|^2)^{1/2} |V(n)| < \infty$. We assume that $\sigma_d(H) = \{e_1 < e_2\}$ with

$$e_1 + n(e_2 - e_1) \notin [0, 4], \quad \forall n \in \mathbb{Z}, \quad (1.2)$$

where $\sigma_d(H)$ is the set of discrete spectrum of H . Further, set ϕ_1, ϕ_2 to be the normalized real valued eigenfunctions associated to e_1, e_2 respectively.

The aim of this paper is to study the long time behavior of small solutions of DNLS (1.1). Before explaining our results, we briefly recall the known results for the “continuous” nonlinear Schrödinger equations (NLS):

$$iu_t = H_c u + |u|^2 u, \quad u : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}.$$

Here, we set $H_c = -\Delta + V$, V is a Schwartz function and assume $\sigma_d(H) = \{e_1 < e_2\}$ with $e_2 < 0$. In this case, it is known that all small solutions decouple into a nonlinear bound state and dispersive wave [20, 48, 49, 50, 51]. Here, a nonlinear bound state is a time periodic solution with the form $e^{-i\omega t} \phi_\omega(x)$ and a dispersive wave is a solution which tends to 0 in L^∞ (or l^∞ in the discrete case) as $t \rightarrow \infty$. In particular, since dispersive wave vanishes locally, we see that all solutions locally converges to some nonlinear bound state. Because the linear Schrödinger equation has quasi-periodic solutions such as $a_1 e^{-ie_1 t} \phi_1 + a_2 e^{-ie_2 t} \phi_2$, it is striking that NLS has no small quasi-periodic solutions. The mechanism which prevent the existence of quasi-periodic solution is

due to the interaction between the continuous spectrum and the discrete spectrum. In particular, when the frequencies $e_1 + n(e_2 - e_1)$ hit the continuous spectrum, there is a damping from the discrete spectrum to the continuous spectrum $[0, \infty)$.

We now come back to the discrete case. For the discrete case there is a possibility that the frequencies $e_1 + n(e_2 - e_1)$ never hit the continuous spectrum since the spectrum of the discrete Laplacian is $[0, 4]$. This is assumption (1.2). In this case, there is no nonlinear interaction between the continuous spectrum and the discrete spectrum. Thus, we can expect there may exist a quasi-periodic solution. Indeed, in this paper we show the existence of quasi-periodic solutions in the form $\Psi(z_1, z_2) \sim z_1\phi_1 + z_2\phi_2$, parametrized by small complex parameters z_1, z_2 (see, Theorem 1.3). Using this family of quasi-periodic solutions, we also show that all solutions decouple into this quasi-periodic solution and dispersive wave (Theorem 1.4).

We now prepare some notations to state our results precisely.

- For $p \geq 1$, $\sigma \in \mathbb{R}$, we set $l^{p,\sigma}(\mathbb{Z}) := \{u = \{u(n)\}_{n \in \mathbb{Z}} \mid \|u\|_{l^{p,\sigma}}^p := \sum_{n \in \mathbb{Z}} \langle n \rangle^{p\sigma} |u(n)|^p < \infty\}$, where $\langle n \rangle := (1 + n^2)^{1/2}$. Further, $l^p(\mathbb{Z}) := l^{p,0}(\mathbb{Z})$.
- We define the inner-product of $l^2(\mathbb{Z})$ by $\langle u, v \rangle := \operatorname{Re} \sum_{n \in \mathbb{Z}} u(n) \overline{v(n)}$.
- For $a \in \mathbb{R}$, we set $l_e^a(\mathbb{Z}) := \{u = \{u(n)\}_{n \in \mathbb{Z}} \mid \|u\|_{l_e^a}^2 := \sum_{n \in \mathbb{Z}} e^{2a|n|} |u(n)|^2 < \infty\}$.
- We often write $a \lesssim b$ by meaning that there exists a constant C s.t. $a \leq Cb$. If we have $a \lesssim b$ and $b \lesssim a$, we write $a \sim b$.
- For a Banach space X equipped with the norm $\|\cdot\|_X$, we set $B_X(\delta) := \{u \in X \mid \|u\|_X < \delta\}$.
- For Banach spaces X, Y , we set $\mathcal{L}(X; Y)$ to be the Banach space of all bounded operators from X to Y , and $\mathcal{L}(X) := \mathcal{L}(X; X)$. Further, we set $\mathcal{L}^n(X; Y)$ inductively by $\mathcal{L}^n(X; Y) = \mathcal{L}(X; \mathcal{L}^{n-1}(X; Y))$ and $\mathcal{L}^0(X; Y) = Y$.
- We set $C^\omega(B_X(\delta); Y)$ to be all real analytic functions from $B_X(\delta)$ to Y . By real analytic functions, we mean that $f : B_X(\delta) \rightarrow Y$ can be written as $f(x) = \sum_{n \geq 0} a_n x^n$ with $\sum_{n \geq 0} \|a_n\|_{\mathcal{L}^n(X; Y)} r^n < \infty$ for all $r < \delta$, where $a_n \in \mathcal{L}^n(X; Y)$ and $a_n x^n := a_n(x, x, \dots, x)$.

It is well known that there exist families of nonlinear bound states of (1.1). For the convenience of the readers, we will give the proof in the appendix of this paper.

Proposition 1.1. *Fix $j \in \{1, 2\}$. There exist $a_0 > 0$ and $\delta_0 > 0$ s.t. for all $z \in B_{\mathbb{C}}(\delta_0)$, there exists $\tilde{e}_j \in C^\omega(B_{\mathbb{R}}(\delta_0^2); \mathbb{R})$ and $q_j \in C^\omega(B_{\mathbb{R}}(\delta_0^2); l_e^{a_0}(\mathbb{Z}; \mathbb{R}))$ s.t. $\langle \phi_j, q_j \rangle = 0$ and*

$$\phi_j(z) := z \tilde{\phi}_j(|z|^2) = z (\phi_j + q_j(|z|^2)), \quad (1.3)$$

satisfies

$$(H - E_j(|z|^2)) \phi_j(z) + |\phi_j(z)|^6 \phi_j(z) = 0, \quad (1.4)$$

where $E_j(|z|^2) = e_j + \tilde{e}_j(|z|^2)$. Further, we have $|\tilde{e}_j(|z|^2)| + \|q_j(|z|^2)\|_{l_e^{a_0}} \lesssim |z|^6$.

Remark 1.2. Notice that if ϕ satisfies (1.4), then $e^{-iE_j t} \phi$ is the solution of (1.1).

The first result of this paper is the existence of quasi-periodic solutions of (1.1).

Theorem 1.3. *There exist $a_1 \in (0, a_0)$ and $\delta_1 \in (0, \delta_0)$ s.t. there exist $\psi \in C^\omega(B_{\mathbb{C}^2}(\delta_1); l_e^{a_1}(\mathbb{Z}; \mathbb{C}))$ and $\varepsilon_j \in C^\omega(B_{\mathbb{R}^2}(\delta_1^2); \mathbb{R})$ for $j = 1, 2$, s.t.*

$$\Psi(z_1, z_2) := \phi_1(z_1) + \phi_2(z_2) + \psi(z_1, z_2),$$

is a solution of (1.1) if z_j ($j = 1, 2$) satisfies

$$i\dot{z}_j = (E_j(|z_j|^2) + \varepsilon_j(|z_1|^2, |z_2|^2)) z_j. \quad (1.5)$$

Further, for arbitrary $\theta \in \mathbb{R}$, we have

$$e^{i\theta} \psi(z_1, z_2) = \psi(e^{i\theta} z_1, e^{i\theta} z_2), \quad (1.6)$$

and

$$\|\psi(z_1, z_2)\|_{l_e^{a_1}} \lesssim |z_1| |z_2| (|z_1|^5 + |z_2|^5), \quad (1.7)$$

$$|\varepsilon_j(|z_1|^2, |z_2|^2)| \lesssim |z_{3-j}|^2 (|z_1|^4 + |z_2|^4). \quad (1.8)$$

The second result of this paper is about the asymptotic behavior of small solution of (1.1).

Theorem 1.4. *Assume H is generic (for the definition see Lemma 5.3 of [26]). Then, there exists $\delta_2 \in (0, \delta_1)$ s.t. if $\|u_0\|_{l^2} < \delta_2$, then the solution of (1.1) with $u(0) = u_0$ exists globally in time and there exist $z_j(t) : [0, \infty) \rightarrow \mathbb{C}$, $\rho_{j,+} \in \mathbb{R}_{\geq 0}$ for $j = 1, 2$ and $v_+ \in l^2$ s.t.*

$$\lim_{t \rightarrow \infty} \|u(t) - \Psi(z_1(t), z_2(t)) - e^{it\Delta} v_+\|_{l^2} = 0, \quad \lim_{t \rightarrow \infty} |z_j(t)| = \rho_{j,+}, \quad (j = 1, 2).$$

Further, we have $\|v_+\|_{l^2} + \rho_{1,+} + \rho_{2,+} \lesssim \|u_0\|_{l^2}$.

Remark 1.5. Theorem 1.3 actually holds even if we replace the nonlinearity $|u|^6 u$ to $|u|^{2p} u$ for $p \in \mathbb{N}$. However, for Theorem 1.4, we need $p \geq 3$. For simplicity, we decided only to consider the case $p = 3$. The assumption for H is used for the linear estimates of e^{itH} . See section 5.

As a corollary of Theorem 1.4, we have orbital stability of nonlinear bound states $\phi_j(z)$. Here, for fixed j and z , we say $\phi_j(z)$ is orbitally stable if for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $\|u(0) - \phi_j(z)\|_{l^2}$, then $\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta} \phi_j(z)\|_{l^2} < \varepsilon$.

Corollary 1.6. *Under the assumptions of Theorem 1.4, for $j = 1, 2$ and $|z| \lesssim \delta_2$, $\phi_j(z)$ is orbitally stable.*

Remark 1.7. For the case $e_1 < 0$, one can show $\phi_1(z)$ is trapped by the energy (i.e. $\phi_1(z)$ is a minimizer of the energy (given in (3.2)) under the constraint $\|u\|_{l^2} = \|\phi_1(z)\|_{l^2}$). By a classical argument by Cazenave-Lions [11], one can conclude that $\phi_1(z)$ is orbitally stable. Similarly, if $e_2 > 4$, $\phi_2(z)$ is a maximizer of the energy under the constraint $\|u\|_{l^2} = \|\phi_2(z)\|_{l^2}$, and we can show it is orbitally stable. However, the case $e_1 < e_2 < 0$ is interesting. In this case $\phi_2(z)$ is not trapped by the energy, which means that $\phi_2(z)$ is not a minimizer (nor a maximizer) of energy E under the constraint $\|u\|_{l^2} = \|\phi_2(z)\|_{l^2}$. Therefore, in this case one cannot show the orbital stability by variational methods.

We now recall the known results related to our results on continuous and discrete NLS. There is a long list of papers on asymptotic stability of both large and small nonlinear bound states of NLS [3, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 22, 23, 25, 27, 29, 33, 34, 35, 36, 38, 39, 42, 43, 45, 46, 48, 49, 50, 51].

The asymptotic stability for small nonlinear bound states of NLS was first proved by Soffer-Weinstein [45]. They assume that the Schrödinger operator $H_c = -\Delta + V$ has exactly one eigenvalue and the initial data is small in some weighted space. Later, Gustafson-Nakanishi-Tsai [29] proved the asymptotic stability in the energy space H^1 for the 3 dimensional case. One of the main tool of [29] was the endpoint Strichartz estimate [28, 31] which collapse in the 1 and 2 dimensional cases. For 1 and 2 dimensional cases, Mizumachi [38, 39] prove the asymptotic stability result in the energy space by replacing the endpoint Strichartz estimates to Kato type smoothing estimates. The results [29, 38, 39] tells us that, under the assumption that H_c has exactly one eigenvalue, the dynamics of small solutions of NLS is similar to the linear Schrödinger equation. This is because the solution of linear Schrödinger equation also decomposes to a periodic solution associated to the eigenvalue and dispersive wave associated to the absolutely continuous spectrum.

The situation changes drastically when H_c has more than two eigenvalues. Indeed, in this case there exist quasi-periodic solutions of the linear Schrödinger equation associated to the two eigenvalues of H_c . However, [44] proved that there exists no small quasi-periodic solution of NLS. Further, [48], [50], [51], [49] proved that if H_c has two eigenvalues with $e_1 < 2e_2$, all small solutions in some weighted space decomposes to a nonlinear bound state and dispersive wave. Recently, [20] extended these result to the case H_c has more than two eigenvalues and removed the assumption $e_1 < 2e_2$. See also related results for nonlinear Klein-Gordon equation (NLKG) [4, 21, 47] and nonlinear Dirac equation [24]. The mechanism which prevents the existence of quasi-periodic solutions is the nonlinear interaction between the eigenvalue and the absolutely continuous spectrum. The non-degeneracy condition for such interaction is called Fermi Golden Rule (FGR) which all the above papers assume.

We now turn to the known results of DNLS. For the case that the discrete Schrödinger operator H has only one eigenvalue, [26, 32] proved the asymptotic stability result in the energy space l^2 . See also [40] for asymptotic stability results in weighted space for lower power nonlinearity. This result corresponds to the continuous case. However, for the case H has two eigenvalues with $e_1 < 0 < 4 < e_2$, [16] proved that the ground state (which is $\phi_1(z)$ in Proposition 1.1) is orbitally stable but not asymptotically stable. For the continuous case, ground state is asymptotically stable, so this result shows that in this case the small solution of continuous and discrete NLS has different asymptotic dynamics. As mentioned in [16], the situation that the nonlinear bound state is orbitally stable but not asymptotically stable suggests that there exist quasi-periodic solutions. Indeed, Theorem 1.3 shows that there exists a 2-parameter family of quasi-periodic solutions which bifurcates from the two eigenvalues of H . Note that the fact that the standing wave is not asymptotically stable is a direct consequence of the existence of quasi-periodic solution near standing waves.

Up to here, we have only discussed the nonlinear bound states and quasi-periodic solutions which bifurcate from the eigenvalues of the Schrödinger operator. We note that it is known that there exist different kinds of periodic and quasi-periodic solutions for DNLS (mainly considered in the translation invariant case, i.e. $V \equiv 0$). First, if the nonlinearity is attractive, there exists a nonlinear bound state which can be approximated by the nonlinear bound state of the continuous NLS [2] (See also [5]). Second, in the “anticontinuous limit” (which is the situation we are putting $\varepsilon \ll 1$ in front of Δ), there exists quasi-periodic solutions (See for example [3, 30, 37]). After rescaling, the quasi-periodic solutions of this kind will have large amplitude.

We prove the existence of the quasi-periodic solutions starting from assuming that the quasi-periodic solution can be written as $\sum_{n \in \mathbb{Z}} e^{-i(\mathcal{E}_1 + n(\mathcal{E}_2 - \mathcal{E}_1))t} v_n$, where $\mathcal{E}_j \sim e_j$ and solve (1.1) for each frequency. Notice that the frequencies $\{\mathcal{E}_1 + n(\mathcal{E}_2 - \mathcal{E}_1)\}_{n \in \mathbb{Z}}$ are generated from the two standing waves and the nonlinearity. Further, there is no intersection between these frequencies and the continuous spectrum of H because of (1.2). This assumption is crucial for the existence of quasi-periodic solution. Indeed, for the continuous NLS case, condition (1.2) always fails because the continuous

spectrum is $[0, \infty)$. Then, by the nonlinear interaction, we have a damping from the point spectrum to the continuous spectrum which prevents the existence of the quasi-periodic solutions. By the same reason, we conjecture that for the case H has more than 3 eigenvalues there will be no quasi-periodic solution like

$$\Psi(z_1, z_2, z_3) \sim z_1\phi_1 + z_2\phi_2 + z_3\phi_3.$$

This is because the nonlinear interaction between the point spectrum and absolutely continuous spectrum arises again and there will be a damping.

For the asymptotic stability result Theorem 1.4, we start from a standard modulation argument and adapt the nonlinear coordinate given in [29]. However, since our quasi-periodic solution is not a standing wave, it seems to be difficult to get a simple equations for the modulation parameters in this coordinate. To overcome this difficulty, we use the Darboux theorem which was introduced in [17] and used in [3, 20, 19, 24]. In fact, after changing the coordinates by the Darboux theorem, we get a well decoupled equations (see (4.34), (4.35)) which are easy to analyze. We note that although we have made the change of coordinate with a real analytic regularity, we actually need only C^3 . The real analyticity comes from the real analyticity of the nonlinearity. Therefore, for the asymptotic stability, we do not need real analyticity. However, for the existence of the quasi-periodic solution, we can only handle a polynomial nonlinearity because we have expanded the solution as $\sum_{n \in \mathbb{Z}} e^{-i(\mathcal{E}_0 + n(\mathcal{E}_1 - \mathcal{E}_0))t} v_n$. Further, real analyticity reduces the amount of some computations for the estimate of the derivatives of the coordinate change (see Lemma 4.11). These are the reasons why we have adapted the real analytic framework for the change of coordinate.

We now mention about the difference between the proof of [3] which shows the asymptotic stability of periodic solutions obtained by the anti-continuous limit (which is a large solutions) and the proof of Theorem 1.4. The difference is that [3] uses the normal form argument infinite times (the Birkhoff normal form). For this method, it is necessary to have the analyticity of the nonlinear term for the convergence of the normal form steps. On the other hand, we only use the normal form argument (the Darboux theorem) once. As mentioned before, our argument only requires C^3 regularity for the coordinate change so it is not necessary to have a analytic nonlinearity for the proof of asymptotic stability. However, we need the nonlinearity to be polynomial for the proof of the existence of the quasi-periodic solution.

The paper is organized as follows: In section 2, we prove Theorem 1.3. In section 3, following [29], we set up the nonlinear coordinate. In section 4, we prove the Darboux theorem and rewrite DNLS in the new coordinate, the new system is given in (4.34)-(4.35). In section 5, we introduce the linear estimates which were originally given in [26] and in section 6, we prove Theorem 1.4. In the appendix we give the proof of Proposition 1.1, Lemma 2.4.

2 Proof of Theorem 1.3

In this section, we construct solutions of (1.1) under the following ansatz:

$$\Psi(z_1, z_2) = \phi_1(z_1) + \phi_2(z_2) + \sum_{m \geq 0} (z_1^{m+1} \bar{z}_2^m v_{1m} + \bar{z}_1^m z_2^{m+1} v_{2m}), \quad (2.1)$$

where, v_{1m}, v_{2m} are real valued and $\langle v_{j0}, \phi_j \rangle = 0$ for $j = 1, 2$.

Remark 2.1. Notice that if $v_{jm} = v_{jm}(|z_1|^2, |z_2|^2)$, then we have $\Psi(e^{i\theta} z_1, e^{i\theta} z_2) = e^{i\theta} \Psi(z_1, z_2)$.

Since we want to reduce the problem of construction of quasi-periodic solution to the construction of solution of system elliptic equations, we assume that for $\varepsilon_j \in \mathbb{R}$ given below (see, (2.5)), z_j

($j = 1, 2$) satisfies

$$\mathbf{i}\dot{z}_j = \mathcal{E}_j z_j, \quad (2.2)$$

where $\mathcal{E}_j = E_j(|z_j|^2) + \varepsilon_j = e_j + \tilde{e}_j(|z_j|^2) + \varepsilon_j$. Then, we have

$$\begin{aligned} \mathbf{i}\partial_t \Psi(z_1, z_2) &= \sum_{j=1,2} \mathcal{E}_j z_j \left(\tilde{\phi}_j(|z_j|^2) + v_{j0} \right) \\ &\quad + \sum_{m \geq 1} z_1^{m+1} \bar{z}_2^m ((m+1)\mathcal{E}_1 - m\mathcal{E}_2) v_{1m} + \bar{z}_1^m z_2^{m+1} ((m+1)\mathcal{E}_2 - m\mathcal{E}_1) v_{2m}, \\ H\Psi(z_1, z_2) &= \sum_{j=1,2} z_j \left(H\tilde{\phi}_j(|z_j|^2) + H v_{j0} \right) + \sum_{m \geq 1} (z_1^{m+1} \bar{z}_2^m H v_{1m} + \bar{z}_1^m z_2^{m+1} H v_{2m}), \end{aligned}$$

where $\tilde{\phi}_j(|z_j|^2)$ is given in (1.3). Further, for $\mathbf{v} = \{v_{j,m}\}_{j=1,2,m \geq 0}$, we have

$$|\Psi(z_1, z_2)|^6 \Psi(z_1, z_2) = |\phi_1(z_1)|^6 \phi_1(z_1) + |\phi_2(z_2)|^6 \phi_2(z_2) + \mathcal{N}(|z_1|^2, |z_2|^2, \mathbf{v}),$$

where

$$\mathcal{N}(|z_1|^2, |z_2|^2, \mathbf{v}) = \sum_{m \geq 0} z_1^{m+1} \bar{z}_2^m N_{1m}(|z_1|^2, |z_2|^2, \mathbf{v}) + \sum_{m \geq 0} \bar{z}_1^m z_2^{m+1} N_{2m}(|z_1|^2, |z_2|^2, \mathbf{v}), \quad (2.3)$$

for some $\{N_{jm}\}_{j=1,2,m \geq 0}$ (see Lemma 2.4 below). Therefore, to construct a solution of (1.1) in the form (2.1), it suffices to solve the system of equations of the coefficients of $z_1^{m+1} \bar{z}_2^m$ and $\bar{z}_1^m z_2^{m+1}$. In particular, we solve the system of elliptic equations

$$\begin{aligned} (H - \omega_{jm}) v_{jm} &= \delta_{0m} \varepsilon_j \tilde{\phi}_j(|z_j|^2) + \delta(j) m (\tilde{e}_1(|z_1|^2) - \tilde{e}_2(|z_2|^2) + \varepsilon_1 - \varepsilon_2) v_{jm} \\ &\quad + (\tilde{e}_j(|z_j|^2) + \varepsilon_j) v_{jm} - N_{jm}, \end{aligned} \quad (2.4)$$

where $\delta_{0m} = 1$ if $m = 0$ and 0 otherwise, $\delta(1) = 1$, $\delta(2) = -1$ and $\omega_{1m} = (m+1)e_1 - me_2$ and $\omega_{2m} = (m+1)e_2 - me_1$. Let $Q_j v = \langle v, \phi_j \rangle \phi_j$. By applying Q to (2.4) with $m = 0$, we have

$$\varepsilon_j = \varepsilon_j(|z_1|^2, |z_2|^2, \mathbf{v}) = \langle N_{j0}(|z_1|^2, |z_2|^2, \mathbf{v}), \phi_j \rangle. \quad (2.5)$$

Therefore, it suffices to solve

$$(H - \omega_{jm}) v_{jm} = \delta_{0m} \varepsilon_j q_j + \delta(j) m (\tilde{e}_1 - \tilde{e}_2 + \varepsilon_1 - \varepsilon_2) v_{jm} + (\tilde{e}_j + \varepsilon_j) v_{jm} - (1 - \delta_{0m} Q_j) N_{jm}, \quad (2.6)$$

where ε_j is now given by (2.5).

Since we want to solve the system (2.6) by fixed point argument, we define a function space X_{ar} for $a, r > 0$ by

$$X_{ar} := \{ \mathbf{v} = \{v_{jm}\}_{j=1,2,m \geq 0} \mid v_{jm} \in l_e^a, \|\mathbf{v}\|_{ar} := \sum_{j=1,2,m \geq 0} r^{2m+1} \|v_{jm}\|_{l_e^a} < \infty \}.$$

For $\mathbf{v} = \{v_{jm}\}_{j=1,2,m \geq 0}$, we set

$$\mathcal{P}\mathbf{v} := \{(1 - \delta_{0m} Q_j) v_{jm}\}_{j=1,2,m \geq 0},$$

and define $X_{ar}^c := \mathcal{P}X_{ar}$. We next define the operator \mathcal{A}, \mathcal{B} on X_{ar}^c by

$$\begin{aligned} \mathcal{A}\mathbf{v} &= \{(H - \omega_{jm})^{-1} v_{jm}\}_{j=1,2,m \geq 0}, \\ \mathcal{B}\mathbf{v} &= \{\delta(j) m (H - \omega_{jm})^{-1} v_{jm}\}_{j=1,2,m \geq 0}, \end{aligned}$$

where $\mathbf{v} = \{v_{jm}\}_{j=1,2,m \geq 0}$. Notice that $(H - e_j)$ are invertible on $(1 - Q_j)l_e^a$ (see Lemma A.3).

Lemma 2.2. For sufficiently small $a > 0$, we have $\mathcal{A}, \mathcal{B} \in \mathcal{L}(X_{ar}^c, X_{ar}^c)$.

Proof. As written above, \mathcal{A} and \mathcal{B} are well defined on X_{ar}^c . By Lemma A.1, we have

$$\|(1+m)(H - \omega_{jm})^{-1}v_{jm}\|_{l_e^a} \lesssim \|v_{jm}\|_{l_e^a}.$$

Therefore, we see that \mathcal{A} and \mathcal{B} are bounded on X_{ar}^c . \square

Using the above notations, we can rewrite (2.6) as

$$\mathbf{v} = \Phi(|z_1|^2, |z_2|^2, \mathbf{v}), \quad (2.7)$$

where

$$\begin{aligned} \Phi(|z_1|^2, |z_2|^2, \mathbf{v}) &= \mathcal{A}\mathbf{q}(|z_1|^2, |z_2|^2, \mathbf{v}) + \sum_{l=1,2} (\tilde{e}_l(|z_l|^2) + \varepsilon_l(|z_1|^2, |z_2|^2, \mathbf{v})) \mathbf{1}_l \mathcal{A}\mathbf{v} \\ &\quad + (\tilde{e}_1(|z_1|^2) - \tilde{e}_2(|z_2|^2) + \varepsilon_1(|z_1|^2, |z_2|^2, \mathbf{v}) - \varepsilon_2(|z_1|^2, |z_2|^2, \mathbf{v})) \mathcal{B}\mathbf{v} - \mathcal{APN}(|z_1|^2, |z_2|^2, \mathbf{v}), \end{aligned} \quad (2.8)$$

where $\mathbf{q}(|z_1|^2, |z_2|^2, \mathbf{v}) = \{\delta_{0m}\varepsilon_j(|z_1|^2, |z_2|^2, \mathbf{v})q_j(|z_j|^2)\}_{j=1,2,m \geq 0}$ and $\mathbf{1}_l \mathbf{v} = \{\delta_{lj}v_{jm}\}_{j=1,2,m \geq 0}$.

To express \mathcal{N} , we introduce the following multilinear operator on X_{ar} .

Definition 2.3. For $\mathbf{v}^k = \{v_{jm}^k\}_{j=1,2,m \geq 0}$, $k = 1, 2, 3$, we define $\mathcal{M}(|z_1|^2, |z_2|^2, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) = \{M_{jm}(|z_1|^2, |z_2|^2, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)\}_{j=1,2,m \geq 0}$ by the relation

$$\begin{aligned} \sum_{m \geq 0} (z_1^{m+1} \overline{z_2}^m M_{1m} + \overline{z_1}^m z_2^{m+1} M_{2m}) &= \sum_{m_1 \geq 0} (z_1^{m_1+1} \overline{z_2}^{m_1} v_{1m_1}^1 + \overline{z_1}^{m_1} z_2^{m_1+1} v_{2m_1}^1) \\ &\quad \times \overline{\sum_{m_2 \geq 0} (z_1^{m_2+1} \overline{z_2}^{m_2} v_{1m_2}^2 + \overline{z_1}^{m_2} z_2^{m_2+1} v_{2m_2}^2)} \sum_{m_3 \geq 0} (z_1^{m_3+1} \overline{z_2}^{m_3} v_{1m_3}^3 + \overline{z_1}^{m_3} z_2^{m_3+1} v_{2m_3}^3). \end{aligned} \quad (2.9)$$

We inductively define $\mathcal{M}_{2k+1}(|z_1|^2, |z_2|^2, \mathbf{v}_1, \dots, \mathbf{v}_{2k+1})$ for $k \geq 1$ by $\mathcal{M}_3 = \mathcal{M}$ and

$$\mathcal{M}_{2k+1}(|z_1|^2, |z_2|^2, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) = \mathcal{M}(|z_1|^2, |z_2|^2, \mathbf{v}^1, \mathbf{v}^2, \mathcal{M}_{2k-1}(|z_1|^2, |z_2|^2, \mathbf{v}^3, \dots, \mathbf{v}^{2k+1})),$$

and $\mathcal{M}_{2k+1}(|z_1|^2, |z_2|^2, \mathbf{v}) := \mathcal{M}_{2k+1}(|z_1|^2, |z_2|^2, \mathbf{v}, \dots, \mathbf{v})$.

Lemma 2.4. Let $\delta < r$. Then, we have $\mathcal{M}_{2k+1} \in C^\omega(B_{\mathbb{R}^2}(\delta^2); \mathcal{L}^{2k+1}(X_{ar}; X_{ar}))$ and

$$\sup_{(z_1, z_2) \in B_{\mathbb{C}^2}(\delta)} \|\mathcal{M}_{2k+1}(|z_1|^2, |z_2|^2, \cdot)\|_{\mathcal{L}^{2k+1}(X_{ar}; X_{ar})} \lesssim 1.$$

We prove Lemma 2.4 in the appendix of this paper.

We set $\Phi_l(|z|^2) = \{\delta_{jl}\delta_{0m}\tilde{\phi}_j(|z|^2)\}_{j=1,2,m \geq 0}$. Using \mathcal{M}_7 , we have

$$\begin{aligned} \mathcal{N}(|z_1|^2, |z_2|^2, \mathbf{v}) &= \mathcal{M}_7(|z_1|^2, |z_2|^2, \Phi_1(|z_1|^2) + \Phi_2(|z_2|^2) + \mathbf{v}) \\ &\quad - \mathcal{M}_7(|z_1|^2, |z_2|^2, \Phi_1(|z_1|^2)) - \mathcal{M}_7(|z_1|^2, |z_2|^2, \Phi_2(|z_2|^2)) \end{aligned} \quad (2.10)$$

We set \mathcal{C}_j by $\mathcal{C}_j \mathbf{v} = \langle \phi_j, v_{j0} \rangle$. Then, since $|\mathcal{C}_j \mathbf{v}| \leq \|v_{j0}\|_{l_e^a} \leq r^{-1} \|\mathbf{v}\|_{ar}$, we have $\mathcal{C}_j \in \mathcal{L}(X_{ar}; \mathbb{R})$ with $\|\mathcal{C}_j\|_{\mathcal{L}(X_{ar}; \mathbb{R})} \leq r^{-1}$. Using \mathcal{C}_j , we have

$$\varepsilon_j(|z_1|^2, |z_2|^2, \mathbf{v}) = \mathcal{C}_j \circ \mathcal{N}(|z_1|^2, |z_2|^2, \mathbf{v}). \quad (2.11)$$

Proposition 2.5. *There exists $r_0 > 0$ s.t. for $\delta < r \leq r_0$, we have $\Phi \in C^\omega(B_{\mathbb{R}^2}(\delta^2) \times X_{ar}^c; X_{ar}^c)$. Further, there exists $C_0 > 0$ s.t. for $|z| \leq \delta$, $\Phi(|z_1|^2, |z_2|^2, \cdot)$ is a contraction mapping on $B_{X_{ar}^c}(Cr^7)$.*

Proof. By Proposition 1.1, Lemma 2.4, (2.8), (2.10) and (2.11), we have $\Phi \in C^\omega(B_{\mathbb{R}^2}(\delta) \times X_{ar}^c; X_{ar}^c)$.

Next, notice that since $\|\Phi_j(|z_j|^2)\|_{ar} \lesssim r$, for $\|\mathbf{v}\|_{ar} \leq r$, we have

$$\begin{aligned} \|\mathcal{N}(|z_1|^2, |z_2|^2, \mathbf{v})\|_{ar} &\leq \|\mathcal{M}_7(|z_1|^2, |z_2|^2, \Phi_1(|z_1|^2) + \Phi_2(|z_2|^2) + \mathbf{v})\|_{ar} \\ &\quad + \|\mathcal{M}_7(|z_1|^2, |z_2|^2, \Phi_1(|z_1|^2))\|_{ar} + \|\mathcal{M}_7(|z_1|^2, |z_2|^2, \Phi_2(|z_2|^2))\|_{ar} \lesssim r^7. \end{aligned}$$

Therefore, by (2.11), we have $|\varepsilon(|z_1|^2, |z_2|^2, \mathbf{v})| \lesssim r^6$. Next, by Proposition 1.1, one can show $\|\mathbf{q}(|z_1|^2, |z_2|^2, 0)\|_{ar} \lesssim r^{13}$. Thus, we have

$$\|\Phi(|z_1|^2, |z_2|^2, \mathbf{v})\|_{ar} \lesssim r^7. \quad (2.12)$$

We set $C_0 > 0$ to satisfy $\|\Phi(|z_1|^2, |z_2|^2, 0)\|_{ar} \leq 2C_0r^7$ for all $z \in B_{\mathbb{C}^2}(r/2)$.

Next, by the multilinearity of $\mathcal{M}_7(|z_1|^2, |z_2|^2, \cdot)$, for $\mathbf{v}^1, \mathbf{v}^2 \in B_{X_{ar}^c}(r)$, we have

$$\|\mathcal{N}(|z_1|^2, |z_2|^2, \mathbf{v}^1) - \mathcal{N}(|z_1|^2, |z_2|^2, \mathbf{v}^2)\|_{ar} \lesssim r^6 \|\mathbf{v}^1 - \mathbf{v}^2\|_{ar}.$$

Thus, we also have $|\varepsilon(|z_1|^2, |z_2|^2, \mathbf{v}^1) - \varepsilon(|z_1|^2, |z_2|^2, \mathbf{v}^2)| \lesssim r^5 \|\mathbf{v}^1 - \mathbf{v}^2\|_{ar}$. Combining the above estimates, we have

$$\|\Phi(|z_1|^2, |z_2|^2, \mathbf{v}^1) - \Phi(|z_1|^2, |z_2|^2, \mathbf{v}^2)\|_{ar} \lesssim r^6 \|\mathbf{v}^1 - \mathbf{v}^2\|_{ar}.$$

This implies that for $r \ll 1$, $\Phi(|z_1|^2, |z_2|^2, \cdot)$ is a contraction mapping on $B_{X_{ar}^c}(C_0r^7)$. \square

Theorem 1.3 follows from Proposition 1.1 almost immediately.

Proof of Theorem 1.3. Let $\mathbf{v}(|z_1|^2, |z_2|^2)$ be the solutions of the fixed point problem (2.7). Then, since $\Phi \in C^\omega(B_{\mathbb{R}^2}(\delta^2) \times X_{ar}^c; X_{ar}^c)$, we have $\mathbf{v} \in C^\omega(B_{\mathbb{R}^2}(\delta^2); X_{ar}^c)$. Now, for $\mathbf{v}(|z_1|^2, |z_2|^2) = \{v_{jm}(|z_1|^2, |z_2|^2)\}_{j=1,2,m \geq 0}$, set

$$\psi(z_1, z_2) = \sum_{m \geq 0} z_1^{m+1} \overline{z_2}^m v_{1m} + \overline{z_1}^m z_2^{m+1} v_{2m},$$

and $\varepsilon_j(|z_1|^2, |z_2|^2) = \varepsilon_j(|z_1|^2, |z_2|^2; \mathbf{v}(|z_1|^2, |z_2|^2))$, where ε_j in the r.h.s. is given by (2.11). Then, we have $\psi \in C^\omega(B_{\mathbb{C}^2}(\delta); l_e^a)$ and $\varepsilon_j \in C^\omega(B_{\mathbb{R}^2}(\delta^2); \mathbb{R})$. The gauge property (1.6) is a direct consequence of the form of ψ . Now, since \mathbf{v} is a unique solution of (2.7) in $B_{X_{ar}^c}(Cr^7)$, we see that $\mathbf{v}(|z_1|^2, 0) = \mathbf{v}(0, |z_2|^2) = 0$. Further, we have $\varepsilon(|z_1|^2, 0) = \varepsilon(0, |z_2|^2) = 0$. Thus, the estimates (1.7) and (1.8) follows from Proposition 2.5 and the analyticity. Finally the fact that $\Psi(z_1, z_2) = \phi_1(z_1) + \phi_2(z_2) + \psi(z_1, z_2)$ is a solution of (1.1) under the condition (1.5) follows from the construction of \mathbf{v} . \square

3 Coordinate

In this section, we prepare the standard modulation argument for the proof of Theorem 1.4. We show that for $u \in l^2$ with $\|u\|_{l^2} \ll 1$, there exists z_1, z_2 s.t. $u = \Psi(z_1, z_2) + v$, where v corresponds to the dispersive wave. In particular, we define a “nonlinear continuous space” $\mathcal{H}_c[z_1, z_2]$ (see (3.6)) and show that we can choose z_1, z_2 s.t. $v \in \mathcal{H}_c[z_1, z_2]$ (Lemma 3.1). Further, since we want to fix the space of the dispersive wave, we introduce a map $R[z_1, z_2] : \mathcal{H}_c[0, 0] \rightarrow \mathcal{H}_c[z_1, z_2]$ (Lemma 3.2) so that we can express u as $u = \Psi(z_1, z_2) + R[z_1, z_2]\eta$ for $\eta \in \mathcal{H}_c[0, 0]$. As a conclusion, we obtain a coordinate $(z_1, z_2, \eta) \in \mathbb{C}^2 \times \mathcal{H}_c[0, 0]$ on $B_{l^2}(\delta)$ with $\delta \ll 1$ (Lemma 3.3).

We first explain how to define the “nonlinear continuous space”. Since $\Psi(e^{-i\mathcal{E}_1 t} z_1, e^{-i\mathcal{E}_2 t} z_2)$ is a solution of (1.1) for fixed z_1, z_2 , we have

$$H\Psi + |\Psi|^6\Psi = i \sum_{j=1,2} (\mathcal{E}_j z_{j,I} D_{j,R} \Psi(z_1, z_2) - \mathcal{E}_j z_{j,R} D_{j,I} \Psi(z_1, z_2)), \quad (3.1)$$

where $D_{j,A} f := \partial_{z_{j,A}} f$ for $j = 1, 2$, $A = R, I$.

Recall that (1.1) conserves the energy E and the l^2 norm, where

$$E(u) = \frac{1}{2} \langle Hu, u \rangle + \frac{1}{8} \langle |u|^6 u, u \rangle. \quad (3.2)$$

Substituting, $\Psi(z_1, z_2) + v$, we have

$$\begin{aligned} E(\Psi + v) &= E(\Psi(z_1, z_2)) + E(v) + \langle H\Psi(z_1, z_2) + |\Psi|^6\Psi, v \rangle + N(z_1, z_2, v) \\ &= E(\Psi(z_1, z_2)) + E(v) + \sum_{j=1,2} (\mathcal{E}_j z_{j,I} \langle iD_{j,R} \Psi, v \rangle - \mathcal{E}_j z_{j,R} \langle iD_{j,I} \Psi, v \rangle) + N(z_1, z_2, v), \end{aligned} \quad (3.3)$$

where we have used (3.1) in the second equality and

$$N(z_1, z_2, v) = \sum_{k=2}^7 \sum_{i+j=k, i \geq j} \langle G_{k,i,j}(z_1, z_2), v^i \bar{v}^j \rangle, \quad (3.4)$$

$$G_{k,i,j}(z_1, z_2) = \sum_{l+r=8-k} C_{k,i,j,l,r} \Psi(z_1, z_2)^l \overline{\Psi(z_1, z_2)}^r, \quad (3.5)$$

for some $C_{k,i,j,l,r} \in \mathbb{R}$. We take the orthogonality condition for v to eliminate the first order term of v in (3.3). Therefore, we set

$$\mathcal{H}_c[z_1, z_2] := \{v \in l^2 \mid \langle iv, D_{j,A} \Psi \rangle = 0, j = 1, 2, A = R, I\}. \quad (3.6)$$

Now, by a standard argument using implicit function theorem, one can choose z_1, z_2 to have $v \in \mathcal{H}_c[z_1, z_2]$. In the following, we use the notation $\phi_{j,R} := \phi_j$ and $\phi_{j,I} = i\phi_j$ for $j = 1, 2$.

Lemma 3.1. *There exists $\delta > 0$ s.t. there exists $(z_1(\cdot), z_2(\cdot)) \in C^\omega(B_{l^2}(\delta); \mathbb{C} \times \mathbb{C})$, s.t.*

$$v(u) := u - \Psi(z_1(u), z_2(u)) \in \mathcal{H}_c[z_1(u), z_2(u)]. \quad (3.7)$$

Proof. Set

$$\mathcal{F}(u, z_1, z_2) := \begin{pmatrix} \langle i(u - \Psi(z_1, z_2)), D_{0,R} \Psi(z_1, z_2) \rangle \\ \langle i(u - \Psi(z_1, z_2)), D_{0,I} \Psi(z_1, z_2) \rangle \\ \langle i(u - \Psi(z_1, z_2)), D_{1,R} \Psi(z_1, z_2) \rangle \\ \langle i(u - \Psi(z_1, z_2)), D_{1,I} \Psi(z_1, z_2) \rangle \end{pmatrix}$$

By implicit function theorem, to obtain $z_1(u), z_2(u)$ which satisfy $\mathcal{F}(u, z_1(u), z_2(u)) = 0$, it suffices to show $\frac{\partial \mathcal{F}}{\partial(z_1, R, z_1, I, z_2, R, z_2, I)} \Big|_{(u, z_1, z_2)=0}$ is invertible if $\|u\|_{l^2} \ll 1$. Since for $j, k = 1, 2$, $A, B = R, I$, $\Psi = o(1)$, $D_{j,A} \Psi = \phi_{j,A} + o(1)$ and $D_{j,A} D_{k,B} \Psi = o(1)$ as $\|u\|_{l^2}, |z_1|, |z_2| \rightarrow 0$, we have

$$\frac{\partial \mathcal{F}}{\partial(z_1, R, z_1, I, z_2, R, z_2, I)} \Big|_{(u, z_1, z_2)=(0,0,0)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Therefore, there exists $(z_1(\cdot), z_2(\cdot)) \in C^\omega(B_{l^2}(\delta); \mathbb{C} \times \mathbb{C})$ s.t. $\mathcal{F}(u, z_1(u), z_2(u)) = 0$ which is equivalent to (3.7). \square

Next, set

$$P_d := \sum_{j=1,2,A=R,I} \langle \cdot, \phi_{j,A} \rangle \phi_{j,A}, \quad P_c := 1 - P_d.$$

Notice that $\mathcal{H}_c[0,0] = P_c l_c^2 =: l_c^2$.

We now define the inverse of the map $P_c|_{\mathcal{H}_c[z_1,z_2]}$ which was used in [29].

Lemma 3.2. *There exists $\delta > 0$ s.t. there exists $\alpha_{j,A} \in C^\omega(B_{\mathbb{C}^2}(\delta); l_e^{a_1}(\mathbb{Z}; \mathbb{C}))$ ($j = 1, 2, A = R, I$), where, a_1 is the constant given in Theorem 1.3, s.t.*

$$\|\alpha_{j,A}(z_1, z_2)\|_{l_e^{a_1}} \lesssim |z|^6,$$

Further,

$$R[z_1, z_2]\eta = \eta + \sum_{j=1,2,A=R,I} \langle \alpha_{j,A}(z_1, z_2), \eta \rangle \phi_{j,A}. \quad (3.8)$$

satisfies $R[z_1, z_2] : l_c^2 \rightarrow \mathcal{H}_c[z_1, z_2]$ and $P_c|_{\mathcal{H}_c[z_1,z_2]} = R[z_1, z_2]^{-1}$.

Proof. We define $\beta_{j,A}[z_1, z_2]\eta \in \mathbb{R}$ ($j = 1, 2, A = R, I$) for $\eta \in l_c^2$ to be the unique solution of

$$\left\langle i \left(\eta + \sum_{j=1,2,A=R,I} (\beta_{j,A}[z_1, z_2]\eta) \phi_{j,A} \right), D_{k,B}\Psi(z_1, z_2) \right\rangle = 0, \quad (3.9)$$

for $k = 0, 1, B = R, I$ and set

$$R[z_1, z_2]\eta = \eta + \sum_{j=1,2,A=R,I} (\beta_{j,A}(z_1, z_2)\eta) \phi_{j,A}$$

By the form of $R[z_1, z_2]$, it is obvious that $P_c R[z_1, z_2] = \text{id}_{l_c^2}$. On the other hand for $\eta \in \mathcal{H}_c[z_1, z_2]$, we have

$$\begin{aligned} R[z_1, z_2]P_c\eta &= P_c\eta + \sum_{j=1,2,A=R,I} (\beta_{j,A}(z_1, z_2)P_c\eta) \phi_{j,A} \\ &= \eta + \sum_{j=1,2,A=R,I} ((\beta_{j,A}(z_1, z_2)P_c\eta) - \langle \eta, \phi_{j,A} \rangle) \phi_{j,A}. \end{aligned}$$

Since $P_c\eta + \sum_{j=1,2,A=R,I} \langle \eta, \phi_{j,A} \rangle \phi_{j,A} \in \mathcal{H}_c[z_1, z_2]$, by the uniqueness of the solution of (3.9), we have

$$\beta_{j,A}(z_1, z_2)P_c\eta = \langle \eta, \phi_{j,A} \rangle, \quad j = 1, 2, A = R, I.$$

Therefore, we have $R[z_1, z_2]P_c\eta = \eta$.

We finally prove (3.9) has a unique solution. (3.9) can be written as

$$\sum_{j=1,2,A=R,I} (\beta_{j,A}(z_1, z_2)\eta \langle \phi_{j,A}, D_{k,B}\Psi \rangle) = -\langle i\eta, D_{k,B}\Psi \rangle = -\langle i\eta, D_{k,B}(q_0 + q_1 + \psi) \rangle, \quad (3.10)$$

where $k = 0, 1$ and $B = R, I$. Writing (3.10) in the matrix form, one can see the coefficient matrix becomes invertible. Therefore, we have a unique solution of (3.9) and the solution $\beta_{j,A}[z_1, z_2]\eta$ can be expressed as $\langle \alpha_{j,A}(z_1, z_2), \eta \rangle$ where $\alpha_{j,A}(z_1, z_2)$ are linear combinations of $D_{k,B}(q_0 + q_1 + \psi)$ for $k = 0, 1$ and $B = R, I$. This expression combined with Theorem 1.3 gives us the desired estimates for $\alpha_{j,A}$ for $j = 1, 2$ and $A = R, I$. \square

Combining Lemmas 3.1, 3.2, we obtain a system of coordinates near the origin of l^2 .

Lemma 3.3. *Let $\delta > 0$ sufficiently small. Then there exists a C^ω diffeomorphism*

$$B_{\mathbb{C}^2 \times l_c^2}(\delta) \ni (z_1, z_2, \eta) \mapsto u = \Psi(z_1, z_2) + R[z_1, z_2]\eta \in l^2. \quad (3.11)$$

Further, we have

$$|z_1| + |z_2| + \|\eta\|_{l^2} \sim \|u\|_{l^2}. \quad (3.12)$$

In the following, we set $(z_1(u), z_2(u), \eta(u)) \in \mathbb{C} \times \mathbb{C} \times l_c^2$ to be the inverse of the map (3.11).

4 Darboux theorem

In the previous section, we have introduced a coordinate (z_1, z_2, η) to express small l^2 functions. Since, l^2 is conserved by the flow of (1.1), we can study the dynamics of small solutions of (1.1) in this coordinate. Indeed, since we have the equation (1.1) and four orthogonal conditions (3.6), (1.1) becomes a system of one discrete evolution equation and four ODEs. However, due to the complexity of the quasi-periodic solution itself, it seems to be difficult to handle this system directly. Therefore, following [20], we make a change of coordinate to have a "canonical" coordinate system and moreover have a simple system of equations (4.34)-(4.35), given in the end of this section.

In the following, we introduce exterior derivatives and symplectic forms.

Definition 4.1 (Exterior derivative). Let $F \in C^\infty(l^2; \mathbf{B})$, where \mathbf{B} be a Banach space (in particular we are considering the case $\mathbf{B} = \mathbb{R}, \mathbb{C}, l^2$). We think F as a 0-form and define its exterior derivative $dF(u)$ (which is a 1-form) by $dF(u) = DF(u)$, where $DF(u)$ is the Fréchet derivative of F . Next, let $\omega(u)$ be 1-form. Then, we define its exterior derivative $d\omega(u)$ (which is a 2-form) by

$$d\omega(u)(X, Y) = \mathcal{L}_X \omega(u)(Y) - \mathcal{L}_Y \omega(u)(X), \quad (4.1)$$

where \mathcal{L}_X is the Lie derivative (i.e. $\mathcal{L}_X \omega(u)(Y) = \frac{d}{d\varepsilon} \big|_{\varepsilon=0} \omega(u + \varepsilon X)(Y)$).

Remark 4.2. In general, for the definition of the exterior derivative, we have to add $-\omega(\mathcal{L}_X(Y))$ to (4.1). However, our space l^2 is flat and we only have to consider constant vector fields for the definition, we can define $d\omega$ as (4.1). See section 6.4 of [1].

We set the symplectic form Ω associated to (1.1) by

$$\Omega(X, Y) := \langle iX, Y \rangle,$$

and

$$B(u)X := \frac{1}{2}\Omega(u, X).$$

Then,

$$dB(u)(X, Y) = \mathcal{L}_X B(u)Y - \mathcal{L}_Y B(u)X = \frac{1}{2}\Omega(X, Y) - \frac{1}{2}\Omega(Y, X) = \Omega(X, Y). \quad (4.2)$$

Therefore, we have $dB(u) = \Omega$.

Next, we introduce a new symplectic form Ω_0 .

Definition 4.3. We define the 1-form B_0 and 2-form Ω_0 by the following.

$$\begin{aligned} B_0(u)X &:= \frac{1}{2}\Omega(\Psi(z_1, z_2), d\Psi(z_1, z_2)(X)) + \frac{1}{2}\Omega(\eta, d\eta(X)), \\ \Omega_0(X, Y) &= \Omega(d\Psi(z_1, z_2)(X), d\Psi(z_1, z_2)(Y)) + \Omega(d\eta(X), d\eta(Y)). \end{aligned}$$

Remark 4.4. As (4.2), we see $dB_0(u) = \Omega_0$.

Remark 4.5. The original symplectic form Ω do not depend on u . However, the new symplectic form Ω_0 depends on u . So, $\Omega_0(X, Y)$ should be written as $\Omega_0(u)(X, Y)$. However, we omit u since there should be no confusion.

Our aim is to change the coordinate system (z_1, z_2, η) to have the new symplectic form Ω_0 . To do so, we use the Moser's argument. Let Γ s.t. $\Omega - \Omega_0 = d\Gamma$ and \mathcal{X}^s satisfies $i_{\mathcal{X}^s}(\Omega_0 + s(\Omega - \Omega_0)) = -\Gamma$, where $i_X\omega(Y) = \omega(X, Y)$. Then, if we set \mathcal{Y}_s to be the solution map of $\frac{d}{ds}\mathcal{Y}_s = \mathcal{X}^s(\mathcal{Y}_s)$, we have

$$\frac{d}{ds}(\mathcal{Y}_s^*\Omega_s) = \mathcal{Y}_s^*(\mathcal{L}_{\mathcal{X}^s}\Omega_s + \partial\Omega_s) = \mathcal{Y}_s^*(di_{\mathcal{X}^s}\Omega_s + d\Gamma) = 0, \quad (4.3)$$

where $\Omega_s = \Omega_0 + s(\Omega - \Omega_0)$. Thus, we have the desired change of coordinate $\mathcal{Y} = \mathcal{Y}_1$ which satisfies $\mathcal{Y}^*\Omega = \Omega_0$. By this argument, it may look like we have already have the change of the coordinate. However, for the application to the asymptotic stability of the quasi-periodic solution, we need have an estimate of \mathcal{Y} in some weighted space.

In the following we construct Γ and \mathcal{X}^s directly.

Lemma 4.6. *Let $\delta > 0$ sufficiently small. Then, there exists $F_\eta \in C^\omega(B_{\mathbb{C}^2 \times P_{cl_e} l_e^{-a_1}}(\delta); l_e^{a_1})$ and $F_{j,A} \in C^\omega(B_{\mathbb{C}^2 \times P_{cl_e} l_e^{-a_1}}(\delta); l_e^{a_1})$ ($j = 1, 2, A = R, I$) s.t. there exists C s.t.*

$$B(u) - B_0(u) - dC = \sum_{j=1,2,A=R,I} \langle F_{j,A}, \eta \rangle dz_{j,A} + \langle F_\eta, d\eta \rangle =: \Gamma.$$

Further, for $j = 1, 2, A = R, I$, we have

$$\|F_\eta\|_{l_e^{a_1}} \lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}, \quad \|F_{j,A}\|_{l_e^{a_1}} \lesssim |z|^6. \quad (4.4)$$

Proof. In the following, we write $\Sigma_{j=1,2,A=R,I}$ as $\Sigma_{j,A}$. Further $\Sigma_{k,B}$ and $\Sigma_{l,C}$ will have the same meaning. First, since

$$\begin{aligned} 2B(u) &= \Omega(u, du) \\ &= \Omega(\Psi + \eta + \sum_{j,A} \langle \alpha_{j,A}, \eta \rangle \phi_{j,A}, d(\Psi + \eta + \sum_{k,B} \langle \alpha_{k,B}, \eta \rangle \phi_{k,B})) \\ &= \Omega(\Psi, d\Psi) + \Omega(\eta, d\eta) + \Omega(\Psi, d\eta) + \Omega(\eta, d\Psi) \\ &\quad + \sum_{k,B} \Omega(\Psi, \phi_{k,B}) d(\langle \alpha_{k,B}, \eta \rangle) + \sum_{j,A} \sum_{k,B} \Omega(\phi_{j,A}, \phi_{k,B}) \langle \alpha_{a,A}, \eta \rangle d(\langle \alpha_{k,B}, \eta \rangle). \end{aligned}$$

So, we have

$$\begin{aligned} 2(B(u) - B_0(u)) &= \Omega(\Psi, d\eta) + \Omega(\eta, d\Psi) + \sum_{k,B} \Omega(\Psi, \phi_{k,B}) d(\langle \alpha_{k,B}, \eta \rangle) \\ &\quad + \sum_{j,A} \sum_{k,B} \Omega(\phi_{j,A}, \phi_{k,B}) \langle \alpha_{a,A}, \eta \rangle d(\langle \alpha_{k,B}, \eta \rangle). \end{aligned} \quad (4.5)$$

The first and second term of r.h.s. of (4.5) can be rewritten as

$$\Omega(\Psi, d\eta) + \Omega(\eta, d\Psi) = d\Omega(\Psi, \eta) + 2\Omega(\eta, d\Psi). \quad (4.6)$$

The third term of r.h.s. of (4.5) can be rewritten as

$$\sum_{k,B} \Omega(\Psi, \phi_{k,B}) d(\langle \alpha_{k,B}, \eta \rangle) = d \left(\sum_{k,B} \Omega(\Psi, \phi_{k,B}) \langle \alpha_{k,B}, \eta \rangle \right) + \sum_{k,B} \langle \alpha_{k,B}, \eta \rangle \Omega(\phi_{k,B}, d\Psi). \quad (4.7)$$

The last term of (4.5) can be rewritten as

$$\begin{aligned} & \sum_{j,A} \sum_{k,B} \Omega(\phi_{j,A}, \phi_{k,B}) \langle \alpha_{a,A}, \eta \rangle d(\langle \alpha_{k,B}, \eta \rangle) \\ &= \sum_{j,A} \sum_{k,B} \Omega(\phi_{j,A}, \phi_{k,B}) \langle \alpha_{a,A}, \eta \rangle (\langle \eta, d\alpha_{k,B} \rangle + \langle \alpha_{k,B}, d\eta \rangle). \end{aligned} \quad (4.8)$$

Combining (4.6), (4.7) and (4.8), we have

$$\begin{aligned} 2(B(u) - B_0(u)) &= d \left(\Omega(\Psi, \eta) + \sum_{k,B} \Omega(\Psi, \phi_{k,B}) \langle \alpha_{k,B}, \eta \rangle \right) \\ &+ \Omega \left(2\eta + \sum_{k,B} \langle \alpha_{k,B}, \eta \rangle \phi_{k,B}, d\Psi \right) + \sum_{l,C} \sum_{k,B} \Omega(\phi_{l,C}, \phi_{k,B}) \langle \alpha_{l,C}, \eta \rangle (\langle \eta, d\alpha_{k,B} \rangle + \langle \alpha_{k,B}, d\eta \rangle). \end{aligned} \quad (4.9)$$

Since $d\Psi = \sum_{j,A} D_{j,A} \Psi dz_{j,A}$, $d\alpha_{k,B} = \sum_{j,A} D_{j,A} \alpha_{k,B} dz_{j,A}$ and $\Omega(\eta, D_{j,A} \Psi) = \Omega(\eta, D_{j,A}(q_0 + q_1 + \psi))$ because $P_c \eta = \eta$, from (4.9), we have

$$B(u) - B_0(u) = dC + \langle F_{j,A}, \eta \rangle dz_{j,A} + \langle F, d\eta \rangle, \quad (4.10)$$

where

$$\begin{aligned} C &= \frac{1}{2} \left(\Omega(\Psi, \eta) + \sum_{k,B} \Omega(\Psi, \phi_{k,B}) \langle \alpha_{k,B}, \eta \rangle \right), \\ F_{j,A} &= -i D_{j,A}(q_0 + q_1 + \psi) + \frac{1}{2} \sum_{k,B} \Omega(\phi_{k,B}, D_{j,A} \Psi) \alpha_{k,B} \end{aligned} \quad (4.11)$$

$$\begin{aligned} &+ \frac{1}{2} \sum_{k,B} \sum_{l,C} \Omega(\phi_{k,B}, \phi_{l,C}) \langle \eta, D_{j,A} \alpha_{l,C} \rangle \alpha_{k,B}, \\ F_\eta &= \sum_{k,B} \sum_{l,C} \Omega(\phi_{k,B}, \phi_{l,C}) \langle \alpha_{k,B}, \eta \rangle \alpha_{l,C} \end{aligned} \quad (4.12)$$

The estimates of (4.4) follows from (4.11) and (4.12) and Lemma 3.2. \square

By lemma 3.12, we have

$$\Omega - \Omega_0 = d(B(u) - B_0(u)) = d(dC + \Gamma) = d\Gamma.$$

We set

$$\Omega_s = \Omega_0 + s(\Omega - \Omega_0),$$

and try to find a solution \mathcal{X}^s of the equation $i\mathcal{X}^s \Omega_s = -\Gamma$.

Lemma 4.7. *Let $\delta > 0$ sufficiently small. Then, there exist $\mathcal{X}_\eta^s \in C^\omega(B_{\mathbb{C}^2 \times P_e l_e^{-a_1}}(\delta); l_e^{a_1})$ and $\mathcal{X}_{j,A}^s \in C^\omega(B_{\mathbb{C}^2 \times P_e l_e^{-a_1}}(\delta); \mathbb{R})$ for $j = 1, 2$, $A = R, I$ s.t. $\mathcal{X}^s := \sum_{j,A} \mathcal{X}_{j,A}^s \partial_{z_{j,A}} + \mathcal{X}_\eta^s \nabla_\eta$ satisfies $i_{\mathcal{X}^s} \Omega_s = -\Gamma$. Further, we have*

$$\|\mathcal{X}_\eta^s\|_{l_e^{a_1}} + \sum_{j,A} |\mathcal{X}_{j,A}^s| \lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}.$$

Proof. We directly solve

$$\Omega_0(\mathcal{X}^s, \cdot) + s(\Omega(\mathcal{X}^s, \cdot) - \Omega_0(\mathcal{X}^s, \cdot)) = -\Gamma. \quad (4.13)$$

In the following, we omit the summation over $j = 1, 2$, $A = R, I$, etc. and j, k, l, r will always be 1, 2 and A, B, C, D will be R, I . First, we have

$$\begin{aligned} \Omega_0(\mathcal{X}^s, \cdot) &= \Omega(d\Psi(\mathcal{X}^s), d\Psi) + \Omega(\mathcal{X}_\eta^s, d\eta) \\ &= \Omega(D_{k,B}\Psi, D_{j,A}\Psi) \mathcal{X}_{k,B}^s dz_{j,A} + \Omega(\mathcal{X}_\eta^s, d\eta), \end{aligned}$$

and

$$\begin{aligned} \Omega(\mathcal{X}^s, \cdot) &= \Omega(d(\Psi + \eta + \langle \alpha_{l,C}, \eta \rangle \phi_{l,C})(\mathcal{X}^s), d(\Psi + \eta + \langle \alpha_{r,D}, \eta \rangle \phi_{r,D})) \\ &= \Omega(D_{k,B}\Psi \mathcal{X}_{k,B}^s + \mathcal{X}_\eta^s + \langle D_{k,B}\alpha_{l,C}, \eta \rangle) \phi_{l,C} \mathcal{X}_{k,B}^s + \langle \alpha_{l,C}, \mathcal{X}_\eta^s \rangle \phi_{l,C}, \\ &\quad D_{j,A}\Psi dz_{j,A} + d\eta + \langle D_{j,A}\alpha_{r,D}, \eta \rangle \phi_{r,D} dz_{j,A} + \langle \alpha_{r,D}, d\eta \rangle \phi_{r,D} \\ &= \Omega_0(\mathcal{X}^s, \cdot) + (G_{j,A,k,B} \mathcal{X}_{k,B}^s + \langle G_{j,A,\eta}, \mathcal{X}_\eta^s \rangle) dz_{j,A} - \mathcal{X}_{k,B}^s \langle G_{k,B,\eta}, d\eta \rangle \\ &\quad + \Omega(\phi_{l,C}, \phi_{r,D}) \langle \alpha_{l,C}, \mathcal{X}_\eta^s \rangle \langle \alpha_{r,D}, d\eta \rangle \end{aligned}$$

where

$$\begin{aligned} G_{j,A,k,B} &= \Omega(D_{k,B}\Psi, \phi_{r,D}) \langle D_{j,A}\alpha_{r,D}, \eta \rangle + \langle D_{k,B}\alpha_{l,C}, \eta \rangle \Omega(\phi_{l,C}, D_{j,A}\Psi) \\ &\quad + \langle D_{k,B}\alpha_{l,C}, \eta \rangle \Omega(\phi_{l,C}, \phi_{r,D}) \langle D_{j,A}\alpha_{r,D}, \eta \rangle, \\ G_{j,A,\eta} &= -iD_{j,A}(q_0 + q_1 + \psi) + \Omega(\phi_{l,C}, D_{j,A}\Psi) \alpha_{l,C} + \Omega(\phi_{l,C}, \phi_{r,D}) \langle D_{j,A}\alpha_{r,D}, \eta \rangle \alpha_{l,C}. \end{aligned}$$

Therefore, (4.13) can be written as

$$(\Omega(D_{k,B}\Psi, D_{j,A}\Psi) + sG_{j,A,k,B}) \mathcal{X}_{k,B}^s + \langle G_{j,A,\eta}, \mathcal{X}_\eta^s \rangle = -\langle F_{j,A}, \eta \rangle \quad (4.14)$$

$$i\mathcal{X}_\eta^s + s(-\mathcal{X}_{k,B}^s G_{k,B,\eta} + \Omega(\phi_{l,C}, \phi_{r,D}) \langle \alpha_{l,C}, \mathcal{X}_\eta^s \rangle \alpha_{r,D}) = -F_\eta, \quad (4.15)$$

where F_η and $F_{j,A}$ are given in Lemma 4.6. We first solve (4.15) fixing $|\mathcal{X}_{k,B}^s| \leq 1$. Notice that (4.15) can be rewritten as

$$(1 - \mathcal{A})\mathcal{X}_\eta^s = -is\mathcal{X}_{k,B}^s G_{k,B,\eta} + iF_\eta,$$

where

$$\mathcal{A}\xi = is\Omega(\phi_{l,C}, \phi_{r,D}) \langle \alpha_{l,C}, \xi \rangle \alpha_{r,D}.$$

Since $\|\mathcal{A}\|_{l_e^{a_1} \rightarrow l_e^{a_1}} \lesssim |z|^6$, we have $\|(1 - \mathcal{A})^{-1}\|_{\mathcal{L}(l_e^{a_1})} \lesssim 1$ by Lemma 3.2. Therefore, we have

$$\mathcal{X}_\eta^s = (1 - \mathcal{A})^{-1} (-is\mathcal{X}_{k,B}^s G_{k,B,\eta} + iF_\eta). \quad (4.16)$$

Substituting, (4.16) into (4.14), we have

$$\begin{aligned} & (\Omega(D_{k,B}\Psi, D_{j,A}\Psi) + sG_{j,A,k,B} - \langle G_{j,A,\eta}, \text{is}(1 - \mathcal{A})^{-1}G_{k,B,\eta} \rangle) \mathcal{X}_{k,B}^s \\ &= -\langle F_{j,A}, \eta \rangle - \langle G_{j,A,\eta}, \text{i}(1 - \mathcal{A})^{-1}F_\eta \rangle. \end{aligned} \quad (4.17)$$

Considering $\Omega(D_{k,B}\Psi, D_{j,A}\Psi) + sG_{j,A,k,B} + \langle G_{j,A,\eta}, \text{is}(1 - \mathcal{A})^{-1}G_{\eta,k,B} \rangle$ as a 4×4 matrix, this matrix has the form

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + o(1).$$

Since this matrix is invertible, we have the solution of (4.17). Therefore, we have the solution of (4.13). Further, we have

$$|\mathcal{X}_{k,B}^s| \lesssim \|F_{j,A}\|_{l_e^{a_1}} \|\eta\|_{l_e^{-a_1}} + \|G_{j,A,\eta}\|_{l^2} \|F\|_{l^2} \lesssim |z|^6 \|\eta\|_{l_e^{-a_1}},$$

and

$$\|\mathcal{X}_\eta^s\|_{l_e^{a_1}} \lesssim |\mathcal{X}_{k,B}^s| \|G_{\eta,k,B}\|_{l_e^{a_1}} + \|F\|_{l_e^{a_1}} \lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}.$$

□

We now construct the desired change of coordinate \mathcal{Y} by the flow of \mathcal{X}^s . We consider the following system

$$\frac{d}{ds} r_j(z_1, z_2, \eta; s) = \mathcal{X}_{j,A}^s(z_1 + r_1(z_1, z_2, \eta; s), z_2 + r_2(z_1, z_2, \eta; s), \eta + r_\eta(z_1, z_2, \eta; s)), \quad (4.18)$$

$$\frac{d}{ds} r_\eta(z_1, z_2, \eta; s) = \mathcal{X}_\eta^s(z_1 + r_1(z_1, z_2, \eta; s), z_2 + r_2(z_1, z_2, \eta; s), \eta + r_\eta(z_1, z_2, \eta; s)), \quad (4.19)$$

with $j = 1, 2$ and the initial condition $r_1(0) = 0$, $r_2(0) = 0$ and $r_\eta(0) = 0$.

Lemma 4.8. *Let $\delta > 0$ sufficiently small. Then, there exists*

$$(r_1, r_2, r_\eta) \in C^\omega(B_{\mathbb{C}^2 \times P_c l_e^{-a_1}}(\delta); C([0, 1]; \mathbb{C}^2 \times l_c^2)),$$

s.t. $(r_1(z_1, z_2, \eta; \cdot), r_2(z_1, z_2, \eta; \cdot), r_\eta(z_1, z_2, \eta; \cdot))$ is the solution of system (4.18)–(4.19) and

$$\sum_{j=1,2} |r_j(z_1, z_2, \eta; 1)| + \|r_\eta(z_1, z_2, \eta; 1)\|_{l_e^{a_1}} \lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}. \quad (4.20)$$

Proof. We solve the system (4.18)–(4.19) by implicit function theorem. Let $\delta > 0$ sufficiently small. Let $(x_1, x_2, \xi) \in C([0, 1]; B_{\mathbb{C}^2 \times l_c^2}(\delta))$ and set

$$\begin{aligned} & \Phi(z_1, z_2, \eta, x_1, x_2, \xi)(s) \\ &= (\Phi_0(z_1, z_2, \eta, x_1, x_2, \xi)(s), \Phi_1(z_1, z_2, \eta, x_1, x_2, \xi)(s), \Phi_\eta(z_1, z_2, \eta, x_1, x_2, \xi)(s)), \end{aligned}$$

where

$$\begin{aligned} \Phi_j(z_1, z_2, \eta, x_1, x_2, \xi)(s) &= x_j(s) - \int_0^s \mathcal{X}_j^\tau(z_1 + x_1(\tau), z_2 + x_2(\tau), \eta + \xi(\tau)) d\tau, \quad j = 1, 2, \\ \Phi_\eta(z_1, z_2, \eta, x_1, x_2, \xi)(s) &= \xi(s) - \int_0^s \mathcal{X}_\eta^\tau(z_1 + x_1(\tau), z_2 + x_2(\tau), \eta + \xi(\tau)) d\tau. \end{aligned}$$

Notice that $\Phi \in C^\omega(B_{\mathbb{C}^2 \times P_c l_e^{-a_1}}(\delta) \times B_{C([0,1]; \mathbb{C}^2 \times P_c l_e^{-a_1})}(\delta); C([0,1]; \mathbb{C}^2 \times l_c^2))$. By the estimate of lemma 4.7 and the analyticity of \mathcal{X}_η^s , we have

$$D_\xi \Phi_\eta(0, 0, 0, 0, 0, 0) = \text{id}_{C([0,1]; l_c^2)}.$$

Therefore, there exists $\tilde{r}_\eta(z_1, z_2, \eta, x_1, x_2)$ s.t. $\Phi_\eta(z_1, z_2, \eta, x_1, x_2, r_\eta(z_1, z_2, \eta, x_1, x_2)) = 0$. Repeatedly, we will have $\tilde{r}_2(z_1, z_2, \eta, x_1)$ and $r_1(z_1, z_2, \eta)$ with desired property. Therefore, setting $r_2 = \tilde{r}_2(z_1, z_2, \eta, r_1(z_1, z_2, \eta))$ and $r_\eta(z_1, z_2, \eta) = \tilde{r}_\eta(z_1, z_2, \eta, r_1(z_1, z_2, \eta), \tilde{r}_2(z_1, z_2, \eta))$, we have the solution of (4.18)–(4.19). We now prove (4.20). From

$$r_j(z_1, z_2, \eta; s) = \int_0^s \mathcal{X}_j^\tau(z_1 + r_1(z_1, z_2, \eta; \tau), z_2 + r_2(z_1, z_2, \eta; \tau), \eta + r_\eta(z_1, z_2, \eta; \tau)) d\tau, \quad (4.21)$$

$$r_\eta(z_1, z_2, \eta; s) = \int_0^s \mathcal{X}_\eta^\tau(z_1 + r_1(z_1, z_2, \eta; \tau), z_2 + r_2(z_1, z_2, \eta; \tau), \eta + r_\eta(z_1, z_2, \eta; \tau)) d\tau, \quad (4.22)$$

and Lemma 4.7, setting $A(s) := \sup_{\tau \in [0, s]} (|r_1(\tau)| + |r_2(\tau)| + \|r_\eta(\tau)\|_{l_e^{a_1}})$, we have

$$A(s) \leq (|z|^6 + A(s)^6) (\|\eta\|_{l_e^{-a_1}} + A(s)). \quad (4.23)$$

Combining (4.23) with the fact $A(0) = 0$, we have (4.20). \square

Now, define \mathcal{Y}_s by

$$\mathcal{Y}_s^* z_j = z_j + r_j(z_1, z_2, \eta; s), \quad j = 1, 2, \quad \mathcal{Y}_s^* \eta = \eta + r_\eta(z_1, z_2, \eta; s).$$

Then, \mathcal{Y}_s satisfies

$$\frac{d}{ds} \mathcal{Y}_s = \mathcal{X}^s(\mathcal{Y}_s),$$

which gives us the desired coordinate change by (4.3). Therefore, setting $\mathcal{Y} := \mathcal{Y}_1$, we have

$$\mathcal{Y}^* \Omega = \Omega_0.$$

We set $r_j(z_1, z_2, \eta) := r_j(z_1, z_2, \eta; 1)$ for $j = 1, 2$ and $r_\eta(z_1, z_2, \eta) := r_\eta(z_1, z_2, \eta; 1)$.

Remark 4.9. We will say $(z'_1, z'_2, \eta') := (\mathcal{Y}^* z_1, \mathcal{Y}^* z_2, \mathcal{Y}^* \eta) = (z_1 + r_1, z_2 + r_2, \eta + r_\eta)$ is the "original" coordinate and (z_1, z_2, η) is the "new" coordinate.

We set the pull-back of the energy by K . That is, we set

$$K(z_1, z_2, \eta) = E \circ \mathcal{Y}(z_1, z_2, \eta) = E(z_1 + r_1, z_2 + r_2, \eta + r_\eta).$$

Now, we define the Hamiltonian vector field associated to F with respect to the symplectic form Ω_0 . We define X_F by

$$\Omega_0(X_F, Y) = \langle \nabla F, Y \rangle.$$

Further, set $(X_F)_{j,A} := dz_{j,A} X_F$ for $j = 1, 2$ and $A = R, I$ and $(X_F)_\eta := d\eta(X_F)$. Then, if u is a solution of (1.1), $z_{j,A}$ and η satisfies

$$\dot{z}_{j,A} = (X_K)_{j,A}, \quad j = 1, 2, \quad A = R, I, \quad \dot{\eta} = (X_K)_\eta.$$

We now directly compute $(X_K)_\eta$. By the definition of X_K , we have

$$\Omega_0(X_K, Y) = \sum_{j,A} \sum_{k,B} \Omega(D_{k,B} \Psi, D_{j,A} \Psi)(X_K)_{k,B} dz_{j,A} Y + \Omega((X_K)_\eta, d\eta Y),$$

and

$$\langle \nabla K, Y \rangle = \sum_{j,A} D_{j,A} F dz_{j,A} Y + \langle \nabla_\eta F, d\eta Y \rangle.$$

Therefore, we have

$$\dot{\eta} = (X_K)_\eta = -i \nabla_\eta K. \quad (4.24)$$

We will postpone the computation of $(X_K)_{j,A}$.

Our next task is to compute the pull-back of the energy K . Before computing, we make one observation. The following lemma corresponds to Lemma 4.11 (Cancellation Lemma) of [20].

Lemma 4.10. *Let $\delta > 0$ sufficiently small. Then, for any $(z_1, z_2) \in B_{\mathbb{C}^2}(\delta)$, we have*

$$\nabla_\eta K(z_1, z_2, 0) = 0.$$

Proof. First, notice that if $\eta = 0$, from Lemma 4.8, we have $r_1 = r_2 = 0$ and $r_\eta = 0$. Therefore, the new and original coordinate corresponds in this case.

Next, recall that if we have the initial condition $u(z'_1, z'_2, 0) = \Psi(z'_1, z'_2)$, since Ψ is the quasi-periodic solution, η' will always be 0. Therefore, the new and original coordinate will correspond for all time and further, we will have $\eta = 0$ for all time.

Now, suppose $\nabla_\eta K(z_1, z_2, 0) \neq 0$. Then, from (4.24), we have

$$i \frac{d}{dt} \Big|_{t=0} \eta = \nabla_\eta K(z_1, z_2, 0) \neq 0. \quad (4.25)$$

However, l.h.s. of (4.25) is 0 because $\eta(t) \equiv 0$. Therefore, we have the conclusion. \square

We prepare another lemma before computing K .

Lemma 4.11. *Set*

$$\begin{aligned} \delta \Psi(z_1, z_2, \eta) &:= \Psi(z_1 + r_1(z_1, z_2, \eta), z_2 + r_2(z_1, z_2, \eta)) - \Psi(z_1, z_2), \\ \delta \eta(z_1, z_2, \eta) &:= R[z_1 + r_1(z_1, z_2, \eta), z_2 + r_2(z_1, z_2, \eta)](\eta + r_\eta(z_1, z_2, \eta)) - \eta. \end{aligned}$$

Then, we have

$$\begin{aligned} \|D^{\mathbf{m}} \delta \Psi(z_1, z_2, \eta)\|_{l_e^{a_1}} + \|D^{\mathbf{m}} \delta \eta(z_1, z_2, \eta)\|_{l_e^{a_1}} &\lesssim |z|^{\max(6-|\mathbf{m}|)} \|\eta\|_{l_e^{a_1}}, \\ \|D^{\mathbf{m}} D_\eta \delta \Psi(z_1, z_2, \eta)\|_{\mathcal{L}(l_e^{-a_1}; l_e^{a_1})} + \|D^{\mathbf{m}} D_\eta \delta \eta(z_1, z_2, \eta)\|_{\mathcal{L}(l_e^{-a_1}; l_e^{a_1})} &\lesssim |z|^{\max(6-|\mathbf{m}|)}, \\ \|D^{\mathbf{m}} D_\eta^2 \delta \Psi(z_1, z_2, \eta)(\xi_1, \xi_2)\|_{l_e^{a_1}} &\lesssim |z|^{\max(6-|\mathbf{m}|)} \|\xi_1\|_{l_e^{-a_1}} \|\xi_2\|_{l_e^{-a_2}}, \end{aligned} \quad (4.26)$$

where $\mathbf{m} = (m_1, \dots, m_{|\mathbf{m}|})$ with $m_j \in \{(k, B) \mid k = 1, 2, B = R, I\}$ and $D^{\mathbf{m}} = D_{m_1} \cdots D_{m_{|\mathbf{m}|}}$.

Proof. By the definition of $\delta\Psi$ and $\delta\eta$, we have

$$\begin{aligned}\delta\Psi(z_1, z_2, \eta) &= \sum_{j,A} \int_0^1 D_{j,A} \Psi(z_1 + sr_1, z_2 + sr_2) ds r_{j,A}, \\ \delta\eta(z_1, z_2, \eta) &= r_\eta + \sum_{j,A} \langle \alpha_{j,A}(z_1 + r_1, z_2 + r_2), \eta + r_\eta \rangle \phi_{j,A}.\end{aligned}$$

Combining the above with Lemma 4.8, we have (4.26) with $|\mathbf{m}| = 0$. The estimates for the derivative respect to $D_{j,A}$ and D_η also follows from Lemma 4.8 because of the analyticity. \square

We now compute the expansion of K .

Lemma 4.12. *We have*

$$K(z_1, z_2, \eta) = E(\Psi(z_1, z_2)) + E(\eta) + \mathcal{N}(z_1, z_2, \eta), \quad (4.27)$$

where \mathcal{N} satisfies

$$|\mathcal{N}(z_1, z_2, \eta)| \lesssim (|z|^6 + \|\eta\|_{l^2}^6) \|\eta\|_{l_e^{-a_2}}^2, \quad (4.28)$$

$$|D_{j,A} \mathcal{N}(z_1, z_2, \eta)| \lesssim (|z|^5 + \|\eta\|_{l^2}^5) \|\eta\|_{l_e^{-a_2}}^2, \quad (4.29)$$

$$\|\nabla_\eta \mathcal{N}(z_1, z_2, \eta)\|_{l_e^{a_2}} \lesssim (|z|^6 + \|\eta\|_{l^2}^6) \|\eta\|_{l_e^{-a_2}}, \quad (4.30)$$

for $a_2 = a_1/3$.

Proof. By Taylor expansion, we have

$$\begin{aligned}E(\Psi(z'_1, z'_2)) &= E(\Psi(z_1, z_2)) + \int_0^1 \langle \nabla E(\Psi(z_1, z_2) + s\delta\Psi(z_1, z_2)), \delta\Psi(z_1, z_2) \rangle ds, \\ E(R[z'_1, z'_2]\eta') &= E(\eta) + \int_0^1 \langle \nabla E(\eta + s\delta\eta(z_1, z_2, \eta)), \delta\eta(z_1, z_2, \eta) \rangle ds.\end{aligned}$$

Therefore, by (3.3), we have (4.27), with

$$\begin{aligned}\mathcal{N}(z_1, z_2, \eta) &= \int_0^1 \langle \nabla E(\Psi + s\delta\Psi), \delta\Psi \rangle ds + \int_0^1 \langle \nabla E(\eta + s\delta\eta), \delta\eta \rangle ds \\ &\quad + \sum_{k=2}^7 \sum_{\substack{i+j=k \\ i \geq j}} \sum_{l+r=8-k} C_{k,i,j,l,r} \langle (\Psi + \delta\Psi)^l (\overline{\Psi} + \overline{\delta\Psi})^r, (\eta + \delta\eta)^i (\overline{\eta} + \overline{\delta\eta})^j \rangle.\end{aligned} \quad (4.31)$$

It suffices to estimate each terms of r.h.s. of (4.31). We first estimate the second term of r.h.s. of (4.31).

$$\begin{aligned}\left| \int_0^1 \langle \nabla E(\eta + s\delta\eta), \delta\eta \rangle ds \right| &\leq \int_0^1 \|\nabla E(\eta + s\delta\eta)\|_{l_e^{-a_1}} ds \|\delta\eta\|_{l_e^{a_1}} \lesssim \|\eta\|_{l_e^{-a_1}} \|\delta\eta\|_{l_e^{a_1}} \\ &\lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}^2.\end{aligned}$$

One can also estimate the $D_{j,A}$ derivative of this term in similar manner. We next compute the ∇_η derivative.

$$\begin{aligned} \left\langle \nabla_\eta \int_0^1 \langle \nabla E(\eta + s\delta\eta), \delta\eta \rangle ds, \xi \right\rangle &= \left\langle \nabla_\eta \int_0^1 \langle H\eta + sH\delta\eta + (\eta + s\delta\eta)^4(\bar{\eta} + s\bar{\delta\eta})^3, \delta\eta \rangle ds, \xi \right\rangle \\ &= \langle H\xi, \delta\eta \rangle + \frac{1}{2} \langle HD_\eta\delta\eta(\xi), \delta\eta \rangle + 4 \int_0^1 \langle (\xi + s(D_\eta\delta\eta(\xi))) |\eta + s\delta\eta|^6, \delta\eta \rangle ds \\ &\quad + 3 \int_0^1 \left\langle \left(\bar{\xi} + s\overline{D_\eta\delta\eta(\xi)} \right) |\eta + s\delta\eta|^4 (\eta + s\delta\eta)^2, \delta\eta \right\rangle ds + \int_0^1 \langle \nabla E(\eta + s\delta\eta), D_\eta\delta\eta(\xi) \rangle ds. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|\nabla_\eta \int_0^1 \langle \nabla E(\eta + s\delta\eta), \delta\eta \rangle ds\|_{l_e^{a_1}} &\lesssim \|\delta\eta\|_{l_e^{a_1}} + \|D_\eta\delta\eta\|_{\mathcal{L}(l_e^{-a_1}; l_e^{a_1})} \|\delta\eta\|_{l_e^{a_1}} \\ &\quad + (1 + \|D_\eta\delta\eta\|_{\mathcal{L}(l_e^{-a_1}; l_e^{a_1})}) (\|\eta\|_{l_\infty}^6 + \|\delta\eta\|_{l_\infty}^6) \|\delta\eta\|_{l_e^{a_1}} + \|D_\eta\delta\eta\|_{\mathcal{L}(l_e^{-a_1}; l_e^{a_1})} \int_0^1 \|\nabla E(\eta + s\delta\eta)\|_{l_e^{-a_1}} ds \\ &\lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}. \end{aligned}$$

The third term of (4.31) can be bounded in similar manner. However, for example, the estimate of $\langle \Psi, |\eta|^7 \eta \rangle$, we have

$$|\langle \Psi, |\eta|^7 \eta \rangle| \leq |z| \|\eta\|_{l_\infty}^5 \|\eta\|_{l_e^{a_1/3}}^2.$$

This is why we have to make a_1 smaller and replace $|z|^j$ to $|z|^j + \|\eta\|_{l_2}^j$.

We finally estimate the first term of (4.31). Expanding $\nabla E(\Psi + s\delta\Psi)$, we have

$$\begin{aligned} \int_0^1 \langle \nabla E(\Psi + s\delta\Psi), \delta\Psi \rangle ds &= \langle \nabla E(\Psi), \delta\Psi \rangle + \frac{1}{2} \langle H\delta\Psi, \delta\Psi \rangle \\ &\quad + \int_0^1 \langle |\Psi + s\delta\Psi|^6 (\Psi + \delta\Psi) - |\Psi|^6 \Psi, \delta\Psi \rangle ds. \end{aligned}$$

The last two terms, which has at least two $\delta\Psi$ can be estimated as before. Now, notice that the only possible source of the first order term of η is $\langle \nabla E(\Psi), \delta\Psi \rangle$. However, by Lemma 4.10, for arbitrary $\xi \in l_c^2$, we have

$$0 = \langle \nabla_\eta K(z_1, z_2, 0), \xi \rangle = \langle \nabla_\eta \langle \nabla E(\Psi), \delta\Psi(z_1, z_2, 0) \rangle, \xi \rangle = \langle \nabla E(\Psi), D_\eta \delta\Psi(z_1, z_2, 0)(\xi) \rangle$$

Therefore, by Taylor expansion, we have

$$\begin{aligned} \langle E(\Psi(z_1, z_2)), \delta\Psi(z_1, z_2, \eta) \rangle &= \int_0^1 (1-s) \langle E(\Psi), D_\eta^2 \delta\Psi(z_1, z_2, s\eta)(\eta, \eta) \rangle ds, \\ \langle \nabla_\eta \langle E(\Psi(z_1, z_2)), \delta\Psi(z_1, z_2, \eta) \rangle, \xi \rangle &= \int_0^1 \langle E(\Psi), D_\eta^2 \delta\Psi(z_1, z_2, s\eta)(\eta, \xi) \rangle ds. \end{aligned}$$

Thus, by Lemma 4.11, we have

$$\begin{aligned} |\langle E(\Psi), \delta\Psi \rangle| &\lesssim \sup_{s \in [0,1]} \|D_\eta^2 \delta\Psi(z_1, z_2, s\eta)(\eta, \eta)\|_{l_e^{a_1}} \lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}^2, \\ \|\nabla_\eta \langle E(\Psi), \delta\Psi \rangle\|_{l_e^{a_1}} &\lesssim \sup_{s \in [0,1]} \|D_\eta \delta\Psi(z_1, z_2, s\eta)(\eta, \cdot)\|_{\mathcal{L}(l_e^{-a_1}; l_e^{a_1})} \lesssim |z|^6 \|\eta\|_{l_e^{-a_1}}. \end{aligned}$$

The estimate for $D_{j,A} \langle E(\Psi), \delta\Psi \rangle$ can be obtained by similar manner. Therefore, we have the conclusion. \square

We now try to obtain the equations which z_j satisfies. Set

$$\{F, G\} := dF(u)X_G(u) = \langle \nabla F(u), X_G(u) \rangle = \Omega(X_F(u), X_G(u))$$

for $F : l^2 \rightarrow \mathbb{C}$. Then, if u is a solution of (1.1), we have

$$\frac{d}{dt}F(u) = \{F, K\},$$

Therefore, setting

$$K(z_1, z_2, \eta) = K_0(z_1, z_2) + K_1(z_1, z_2, \eta),$$

where $K_0(z_1, z_2) = E(\Psi(z_1, z_2))$. we have

$$\dot{z}_j = \{z_j, K_0\} + \{z_j, K_1\}.$$

Now, since

$$\Omega_0(X_{K_n}, Y) = dK_n(u)Y = \sum_{j=1,2,A=R,I} \partial_{z_{j,A}} K_n Y_{j,A} + \langle \nabla_\eta K_n, Y_\eta \rangle, ,$$

and

$$\Omega(X_{K_n}, Y) = \sum_{j,k=1,2,A,B=R,I} \Omega(D_{k,B}\Psi(z_1, z_2), D_{j,A}\Psi(z_1, z_2))(X_{K_n})_{k,B}(Y)_{j,A} + \Omega((X_{K_n})_\eta, Y_\eta),$$

we have

$$(X_F)_z = \mathcal{A}(z_1, z_2)^{-1} \partial_z F, \tag{4.32}$$

where,

$$(X_F)_z = \begin{pmatrix} (X_F)_{1,R} \\ (X_F)_{1,I} \\ (X_F)_{2,R} \\ (X_F)_{2,I} \end{pmatrix}, \quad \partial_z F = \begin{pmatrix} D_{1,R}F \\ D_{1,I}F \\ D_{2,R}F \\ D_{2,I}F \end{pmatrix},$$

and

$$\mathcal{A}(z_1, z_2) = \begin{pmatrix} a_{1,R,1,R}(z_1, z_2) & a_{1,I,1,R}(z_1, z_2) & a_{2,R,1,R}(z_1, z_2) & a_{2,I,1,R}(z_1, z_2) \\ a_{1,R,1,I}(z_1, z_2) & a_{1,I,1,I}(z_1, z_2) & a_{2,R,1,I}(z_1, z_2) & a_{2,I,1,I}(z_1, z_2) \\ a_{1,R,2,R}(z_1, z_2) & a_{1,I,2,R}(z_1, z_2) & a_{2,R,2,R}(z_1, z_2) & a_{2,I,2,R}(z_1, z_2) \\ a_{1,R,2,I}(z_1, z_2) & a_{1,I,2,I}(z_1, z_2) & a_{2,R,2,I}(z_1, z_2) & a_{2,I,2,I}(z_1, z_2) \end{pmatrix},$$

where $a_{j,A,k,B}(z_1, z_2) = \Omega(D_{j,A}\Psi(z_1, z_2), D_{k,B}\Psi(z_1, z_2))$. Notice that since

$$\mathcal{A}(z_1, z_2) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + o(1),$$

$\mathcal{A}(z_1, z_2)$ is invertible

We will not compute $\{z_j, K_0\}$ directly but use the fact that $\Psi(z_1, z_2)$ is the solution of (1.1).

Lemma 4.13. *We have*

$$\{z_j, K_0(z_1, z_2)\} = -i\mathcal{E}_j(|z_1|^2, |z_2|^2)z_j$$

Proof. First, if $\eta = 0$, we have $z'_0 = z_0$ and $z'_1 = z_1$. Therefore, since $\Psi(z_1, z_2)$ is the solution (1.1) if $i\dot{z}_j = \mathcal{E}(|z_1|^2, |z_2|^2)z_j$, we have

$$-i\mathcal{E}(|z_1|^2, |z_2|^2)z_j = \dot{z}_j = \{z_j, K(z_1, z_2, \eta)\}_{\eta=0} = \{z_j, K_0(z_1, z_2)\}_{\eta=0} + \{z_j, K_1(z_1, z_2, \eta)\}_{\eta=0}.$$

On the other hand, from (4.32), we see that $\{z_j, K_1(z_1, z_2, \eta)\}_{\eta=0} = 0$ because it consists from the $D_{k,B}$ derivative or K_1 which is 0 if $\eta = 0$. Therefore, we have

$$-i\mathcal{E}(|z_1|^2, |z_2|^2)z_j = \{z_j, K_0(z_1, z_2)\}_{\eta=0}.$$

Finally, since the symplectic form Ω_0 do not depend on η (although it depends on z_j), we have the conclusion. \square

We set $R_j = \{z_j, K_1(z_1, z_2, \eta)\}$. Then, by (4.32), we have $R_j = \{z_j, \mathcal{N}(z_1, z_2, \eta)\}$. Futher, combining (4.32) with (4.29), we have

$$|R_j| \lesssim (|z|^5 + \|\eta\|_{l^2})\|\eta\|_{l_e^{-a_1}}. \quad (4.33)$$

As a conclusion of this section, we have the equations of z_j and η .

$$i\eta_t = H\eta + P_c(|\eta|^6\eta + \nabla_\eta \mathcal{N}), \quad (4.34)$$

$$\dot{z}_j = -i\mathcal{E}(|z_1|^2, |z_2|^2)z_j + R_j, \quad j = 1, 2. \quad (4.35)$$

5 Linear estimates

In this section, we introduce the linear estimates for the proof of Theorem 1.4. Lemmas 5.2–5.5 can be found in [26]. See also [41] and [32]. In the following we always assume H is generic in the sense of Lemma 5.3 of [26].

Definition 5.1. We say the pair of numbers (r, p) is admissible if

$$\frac{2}{r} + \frac{1}{p} = \frac{1}{2}, \quad (r, p) \in [4, \infty] \times [2, \infty].$$

We set

$$X_{r,p} := l^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], l^p)), \quad X'_{r,p} = l^{(\frac{3}{2}r)'}(\mathbb{Z}, L_t^1([n, n+1], l^{p'})),$$

where p' is the Hölder conjugate of p (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$).

Lemma 5.2 (Dispersive estimate). *We have*

$$\|e^{-itH}P_c\|_{\mathcal{L}(l^1; l^\infty)} \lesssim \langle t \rangle^{-1/3}.$$

Lemma 5.3 (Strichartz estimate). *Let (p, r) , (p_1, r_1) and (p_2, r_2) admissible. Then, we have*

$$\|e^{-itH}P_cf\|_{X_{r,p}} \lesssim \|f\|_{l^2},$$

and

$$\left\| \int_0^t e^{-i(t-s)H}P_cg(s)ds \right\|_{X_{r_1,p_1}} \lesssim \|g\|_{X'_{r_2,p_2}}.$$

Lemma 5.4 (Kato Smoothing). *Let $\sigma > 1$. Then, we have*

$$\|e^{-itH}P_c f\|_{L^t l^2, -\sigma} \lesssim \|f\|_{l^2},$$

and

$$\left\| \int_0^t e^{-i(t-s)H} P_c g(s) ds \right\|_{L_t^2 l^2, -\sigma} \lesssim \|g\|_{L^2 l^2, \sigma}.$$

Lemma 5.5. *Let $\sigma > 1$ and (r, p) admissible. Then, we have*

$$\left\| \int_0^t e^{-i(t-s)H} P_c g(s, \cdot) ds \right\|_{X_{r,p}} \lesssim \|g\|_{L_t^2 l^2, \sigma}.$$

6 Proof of Theorem 1.4

We are now in the position to prove Theorem 1.4 and Corollary 1.6.

Fix $\sigma > 1$ and set $X = X_{4,\infty} \cap X_{\infty,2} \cap L^2 l^2, -\sigma$.

Proposition 6.1. *Under the hypothesis of Theorem 1.4, there exists $\epsilon_0 > 0$ s.t. if $\|u_0\|_{l^2} = \epsilon < \epsilon_0$, we have*

$$\|\eta\|_X \lesssim \|\eta(0)\|_{l^2}, \quad (6.1)$$

$$\left\| \frac{d}{dt} |z_j|^2 \right\|_{L^1} \lesssim \epsilon^6 \|\eta(0)\|_{l^2}^2, \quad j = 1, 2. \quad (6.2)$$

Proof. First, by the l^2 conservation of (1.1) and (4.20) of Lemma 4.8, we have

$$\begin{aligned} |z_1| + |z_2| + \|\eta\|_{l^2} &\lesssim \sum_{j=1,2} |z'_j| + |r_j(z_1, z_2, \eta)| + \|\eta\|_{l^2} + \|r_\eta(z_1, z_2, \eta)\|_{l^2} \\ &\lesssim \epsilon + |z|^6 \|\eta\|_{l^2}. \end{aligned}$$

Therefore, we have

$$|z_1| + |z_2| + \|\eta\|_{l^2} \lesssim \epsilon, \quad (6.3)$$

for all time t . By (4.34), for any admissible pair (r, p) , we have

$$\begin{aligned} \|\eta\|_{X_{r,p}} &\leq \|e^{-itH}\eta(0)\|_{X_{r,p}} + \left\| \int_0^t e^{-i(t-s)H} P_c \nabla_\eta \mathcal{N} ds \right\|_{X_{r,p}} + \left\| \int_0^t e^{-i(t-s)H} P_c |\eta|^6 \eta ds \right\|_{X_{r,p}} \\ &\lesssim \|\eta(0)\|_{l^2} + \|\nabla_\eta \mathcal{N}\|_{L^2 l^2, \sigma} + \|\eta\|_{L^1 l^2}^6 \\ &\lesssim \|\eta(0)\|_{l^2} + \epsilon^6 \|\eta\|_{L^2 l^2, -\sigma} + \|\eta\|_{L^\infty l^2} \|\eta\|_{X_{4,\infty}}^6, \end{aligned}$$

where we have used Lemma 5.3 and 5.5 in the first inequality and (4.30) in the second inequality. Again by (4.34) and Lemma 5.3, 5.4, we have

$$\begin{aligned} \|\eta\|_{L^2 l^2, -\sigma} &\lesssim \|\eta(0)\|_{l^2} + \|\nabla_\eta \mathcal{N}\|_{L^2 l^2, \sigma} + \int_0^\infty \|\eta\|^7_{l^2} ds \\ &\lesssim \|\eta(0)\|_{l^2} + \epsilon^6 \|\eta\|_{L^2 l^2, -\sigma} + \|\eta\|_{X_{14/3,14}}^7, \end{aligned}$$

where we have use $\|\eta\|_{L^7 L^{14}} \leq \|\eta\|_{X_{14/3,14}}$ in the second inequality. Therefore, we have

$$\|\eta\|_X \lesssim \|\eta(0)\|_{l^2} + \epsilon^6 \|\eta\|_X + \|\eta\|_X^7.$$

By continuity argument, we have (6.1).

Next, multiplying $\overline{z_j}$ to (4.35) and taking the real part, we have

$$\frac{d}{dt}|z_j|^2 = R_j \overline{z_j} + \overline{R_j} z_j.$$

Therefore, by (4.33), we have

$$\left\| \frac{d}{dt}|z_j|^2 \right\|_{L^1} \leq (\|z_1\|_{L^\infty}^6 + \|z_2\|_{L^\infty}^6 + \|\eta\|_{L^\infty l^2}^6) \|\eta\|_{L^2 l^2, -\sigma}^2 \lesssim \epsilon^6 \|\eta\|_{L^2 l^2, -\sigma}^2.$$

Combining the above with (6.1), we obtain (6.2). \square

We now prove Theorem 1.4.

Proof of Theorem 1.4. By Proposition 6.1, we see that there exists $\rho_{j,+}$ and v_+ s.t.

$$|z_j(t)| \rightarrow \rho_{j,+}, \quad \text{and} \quad \|\eta(t) - e^{it\Delta} \eta_+\|_{l^2} \rightarrow 0.$$

with $\rho_{0,+} + \rho_{1,+} + \|v_+\|_{l^2} \lesssim \epsilon$.

Now, by Lemma 5.2 we have $\|\eta(t)\|_{l^\infty} \rightarrow 0$ for any $a > 0$. Therefore, by Lemma 4.8, we see $\|\eta'(t) - \eta(t)\|_{l^2} \rightarrow 0$ and $||z_j(t)| - |z_j'(t)|| \rightarrow 0$ as $t \rightarrow \infty$. Here, (z_1', z_2', η') are the original coordinates (see remark 4.9). Therefore, we have the conclusion. \square

We next prove Corollary 1.6.

Proof of Corollary 1.6. Fix $j \in \{1, 2\}$ and fix $z_j \in \mathbb{C}$ with $|z_j| \ll 1$. Now, let $0 \ll \epsilon \ll |z_j|$ and assume $\|u(0) - \phi_2(z_j)\|_{l^2} \lesssim \epsilon$, then we have $|z_{3-j}(0)| + |z_j - z_j(0)| + \|\eta(0)\|_{l^2} \lesssim \epsilon$. Further, by Proposition 6.1, we have

$$\sup_{t \geq 0} (|z_{3-j}(t)|^2 + \|\eta(t)\|_{l^2}) \lesssim |z_{3-j}(0)|^2 + \|\eta(0)\|_{l^2}^2,$$

and

$$\sup_{t \geq 0} (|z_j(t)|^2 - |z_j|^2) \lesssim |z_j(0)|^2 - |z_j|^2 + |z_j|^6 \|\eta(0)\|_{l^2}.$$

Therefore, going back to the original coordinate, we have the conclusion. \square

A Proof of Proposition 1.1

In this section, we prove Proposition 1.1. Before proving Proposition 1.1, we prepare an elementary estimate.

Lemma A.1. *Let $\delta > 0$. Then there exists $a(\delta) > 0$ s.t. for $a \in (0, a(\delta))$ and for $\lambda \notin (-\delta, 4 + \delta) \cup (e_1 - \delta, e_1 + \delta) \cup (e_2 - \delta, e_2 + \delta)$, we have*

$$\|(H - \lambda)^{-1}\|_{\mathcal{L}(l_e^a)} \lesssim_\delta \langle \lambda \rangle^{-1}. \quad (\text{A.1})$$

Further, let $j = 1, 2$. Then, for sufficiently small $a > 0$, we have

$$\left\| \left((H - e_j)|_{\phi_j^\perp} \right)^{-1} \right\|_{\mathcal{L}(l_e^a)} \lesssim 1. \quad (\text{A.2})$$

Proof. Set $T_{a,N}$ by

$$(T_{a,N}v)(n) = e^{a \min(|n|, N)} v(n).$$

We first claim there exists $B_{a,N} : l^2 \rightarrow l^2$ s.t. $\|B_{a,N}\|_{l^2 \rightarrow l^2} \lesssim a$ (the implicit constant do not depend on N) and

$$T_{a,N}(H - \lambda)T_{a,N}^{-1} = H - \lambda + B_{a,N}.$$

Indeed, setting $B_{a,N} = T_{a,N}(-\Delta)T_{a,N}^{-1} + \Delta$, we have

$$(B_a u)(n) = \left(1 - e^{a(\min(|n|, N) - \min(|n+1|, N))}\right) u(n+1) + \left(1 - e^{a(\min(|n|, N) - \min(|n-1|, N))}\right) u(n-1).$$

Since $|1 - e^{a(\min(|n|, N) - \min(|n+1|, N))}| \lesssim a$ and $|1 - e^{a(\min(|n|, N) - \min(|n-1|, N))}| \lesssim a$, we have the desired bound for $B_{a,N}$. Now, since

$$\begin{aligned} T_{a,N}(H - \lambda)^{-1}T_{a,N}^{-1} &= (T_{a,N}(H - \lambda)T_{a,N})^{-1} = (H - \lambda + B_{a,N})^{-1} \\ &= (H - \lambda)^{-1}(1 + (H - \lambda)^{-1}B_{a,N})^{-1}. \end{aligned}$$

Therefore, by Neumann expansion and since $\|(H - \lambda)^{-1}\|_{l^2 \rightarrow l^2} \lesssim \delta^{-1}$, if we take $a > 0$ sufficiently small s.t. $a\delta^{-1} \ll 1$, we have

$$\|T_{a,N}(H - \lambda)^{-1}T_{a,N}^{-1}\|_{\mathcal{L}(l^2)} \lesssim \|(H - \lambda)^{-1}\| \lesssim_\delta \lambda^{-1}$$

This implies that for $u \in l_e^a$,

$$\|T_{a,N}(H - \lambda)^{-1}u\|_{l^2} \lesssim_\delta \lambda^{-1} \|T_{a,N}u\|_{l^2} \leq \lambda^{-1} \|u\|_{l_e^a}.$$

Taking $N \rightarrow \infty$, we obtain (A.1).

Next, we prove (A.2). Suppose $u, f \perp \phi_j$ and $(H - e_j)u = f$, $u \in l^2$, $f \in l_e^a$. Set $P := \langle \cdot, \phi_j \rangle \phi_j$ and $Q = 1 - P$. Now, we have

$$T_{a,N}f = (H - e_j + B_{a,N})T_{a,N}u = (H - e_j + B_{a,N})(QT_{a,N}u + \langle u, T_{a,N}\phi \rangle \phi).$$

Therefore, we have

$$(H - e_j)QT_{a,N}u = T_{a,N}f - B_{a,N}QT_{a,N}u - \langle u, T_{a,N}\phi \rangle B_{a,N}\phi.$$

Now, by $f, \phi \in l_e^a$, where $a > 0$ is sufficiently small so that $\phi_j \in l_e^a$, we have

$$\|QT_{a,N}u\|_{l^2} \lesssim \|f\|_{l_e^a} + a\|QT_{a,N}u\|_{l^2} + a\|u\|_{l^2}.$$

Thus, for a sufficiently small,

$$\|QT_{a,N}u\|_{l^2} \lesssim \|f\|_{l_e^a} + a\|u\|_{l^2},$$

and

$$\|T_{a,N}u\|_{l^2} \leq \|QT_{a,N}u\|_{l^2} + \|PT_{a,N}u\|_{l^2} \lesssim \|f\|_{l_e^a} + \|u\|_{l^2}.$$

Finally, taking $N \rightarrow \infty$, we have

$$\|u\|_{l_e^2} \lesssim \|f\|_{l_e^2},$$

where we have used the fact that $\|u\|_{l^2} \lesssim \|f\|_{l^2} \leq \|f\|_{l_e^2}$. \square

We now prove Proposition 1.1.

Proof of Proposition 1.1. For simplicity, we write ϕ_j as ϕ , e_j as e and E_j as E . Consider a solution in the form $z(\phi + q(|z|^2))$ with real valued q with $\langle \phi, q \rangle = 0$. Now, substitute it in the equation and we have

$$Hq + |z|^6|\phi + q|^6(\phi + q) = eq + (E - e)(\phi + q).$$

Then, we have

$$\begin{aligned} (|z|^6|\phi + q|^6(\phi + q), \phi) &= E - e, \\ Hq + Q(|z|^6|\phi + q|^6(\phi + q)) &= Eq. \end{aligned}$$

Therefore, we set

$$E(|z|^2, q) := e + |z|^6 \langle |\phi + q|^6(\phi + q), \phi \rangle, \quad (\text{A.3})$$

and we have

$$\begin{aligned} (H - e)q &= (E(z, q) - e)q - Q(|z|^6|\phi + q|^6(\phi + q)) \\ &= |z|^6(f(q), \phi)q - |z|^6Qf(q), \end{aligned}$$

where $f(q) = |\phi + q|^6(\phi + q)$. We set $\mathcal{F} : Ql_e^a \times \mathbb{R} \rightarrow Ql_e^a$ by

$$\mathcal{F}(q, s) := (H - e)q - s^3(f(q), \phi)q + s^3Qf(q).$$

Then, \mathcal{F} is real analytic with respect to q and s . Further, since

$$D_q\mathcal{F}(q, s)|_{(q,s)=(0,0)} = H - e$$

is invertible in Ql_e^a for sufficiently small $a > 0$, by implicit function theorem, for sufficiently small s , there exists $q(s)$ s.t. $q(s)$ is real analytic with respect to s and $\mathcal{F}(q(s), s) = 0$. Further, comparing the Taylor series of

$$q(s) = s^3(H - e)^{-1}((f(q), \phi) - Qf(q)),$$

we see $\|q(s)\|_{l_e^a} \lesssim s^3$. Therefore, $q(|z|^2)$ is the desired solution. Finally, set $E(s) = E(s, q(s))$, where the r.h.s. is given in (A.3). Then, since $E(s, q)$ and q is both real analytic, $E(s)$ also becomes real analytic. The estimate $|E(|z|^2) - e| \lesssim |z|^6$ also follows from (A.3). \square

B Proof of Lemma 2.4

Proof of Lemma 2.4. Set $\mathbf{v}^k = \{v_{jm}^k\}_{j=1,2,m \geq 0}$, ($k = 1, 2, 3$). Then, using the relation (2.9), we have

$$\begin{aligned}
M_{1m} = & \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} |z_1|^{2l+2} |z_2|^{2l} v_{1m_1}^1 v_{1l}^2 v_{1(l+m-m_1)}^3 + \sum_{l-1 \geq m_2 \geq 0} |z_1|^{2l} |z_2|^{2l} v_{1(m+l)}^1 v_{1m_2}^2 v_{2(l-m_2-1)}^3 \\
& + \sum_{\substack{m_2, m_3 \geq 0 \\ m_2+m_3 \leq m-1}} v_{1(m-m_2-m_3-1)}^1 v_{2m_2}^2 v_{1m_3}^3 + \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} |z_1|^{2l} |z_2|^{2l+2} v_{1m_1}^1 v_{2(l+m-m_1)}^2 v_{2l}^3 \\
& + \sum_{l-1 \geq m_1 \geq 0} |z_1|^{2l} |z_2|^{2l} v_{2m_1}^1 v_{1(l-m_1-1)}^2 v_{1(m+l)}^3 + \sum_{\substack{l \geq 0 \\ l+m \geq m_2 \geq 0}} |z_1|^{2l} |z_2|^{2l+2} v_{2l}^1 v_{2m_2}^2 v_{1(l+m-m_2)}^3 \\
& + \sum_{l \geq m_1 \geq 0} |z_1|^{2l} |z_2|^{2l+4} v_{2m_1}^1 v_{2(m+l+1)}^2 v_{2(l-m_1)}^3,
\end{aligned}$$

and

$$\begin{aligned}
M_{2m} = & \sum_{l \geq m_1 \geq 0} |z_1|^{2l+4} |z_2|^{2l} v_{1m_1}^1 v_{1(m+l+1)}^2 v_{1(l-m_1)}^3 + \sum_{\substack{l \geq 0 \\ m+l \geq m_2 \geq 0}} |z_1|^{2l+2} |z_2|^{2l} v_{1l}^1 v_{1m_2}^2 v_{2(m+l-m_2)}^3 \\
& + \sum_{l-1 \geq m_1 \geq 0} |z_1|^{2l} |z_2|^{2l} v_{1m_1}^1 v_{2(l-m_1-1)}^2 v_{2(m+l)}^3 + \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} |z_1|^{2l+2} |z_2|^{2l} v_{2m_1}^1 v_{1(l+m-m_1)}^2 v_{1l}^3 \\
& + \sum_{\substack{m_2, m_3 \geq 0 \\ m_2+m_3 \leq m+1}} v_{2(m-m_2-m_3-1)}^1 v_{1m_2}^2 v_{2m_3}^3 + \sum_{l-1 \geq m_2 \geq 0} |z_1|^{2l} |z_2|^{2l} v_{2(m+l)}^1 v_{2m_2}^2 v_{1(l-m_2-1)}^3 \\
& + \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} |z_1|^{2l} |z_2|^{2l+2} v_{2m_1}^1 v_{2l}^2 v_{2(l+m-m_1)}^3.
\end{aligned}$$

Therefore, we can express \mathcal{M} such as

$$\begin{aligned}
\mathcal{M}(|z_1|^2, |z_2|^2, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) = & \tilde{\mathbf{m}}^{00}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) + \sum_{l \geq 0} \left(|z_1|^{2(l+2)} |z_2|^{2l} \mathbf{m}^{(l+2)l}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) \right. \\
& + |z_1|^{2(l+1)} |z_2|^{2l} \mathbf{m}^{(l+1)l}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) + |z_1|^{2l} |z_2|^{2l} \mathbf{m}^{ll}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) \\
& \left. + |z_1|^{2l} |z_2|^{2(l+1)} \mathbf{m}^{l(l+1)}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) + |z_1|^{2l} |z_2|^{2(l+2)} \mathbf{m}^{l(l+2)}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) \right),
\end{aligned}$$

where $\tilde{\mathbf{m}}^{00} = \{\tilde{m}_{jm}^{00}\}_{j=1,2,m \geq 0}$ and $\mathbf{m}^{l_1 l_2} = \{m_{jm}^{l_1 l_2}\}_{j=1,2,m \geq 0}$ are given by

$$\tilde{m}_{1m}^{00} = \sum_{\substack{m_2, m_3 \geq 0 \\ m_2+m_3 \leq m-1}} v_{1(m-m_2-m_3-1)}^1 v_{2m_2}^2 v_{1m_3}^3, \quad \tilde{m}_{2m}^{00} = \sum_{\substack{m_2, m_3 \geq 0 \\ m_2+m_3 \leq m+1}} v_{2(m-m_2-m_3-1)}^1 v_{1m_2}^2 v_{2m_3}^3,$$

and

$$\begin{aligned}
m_{1m}^{(l+2)l} &= 0, \quad m_{2m}^{(l+2)l} = \sum_{l \geq m_1 \geq 0} v_{1m_1}^1 v_{1(m+l+1)}^2 v_{1(l-m_1)}^3, \quad m_{1m}^{(l+1)l} = \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} v_{1m_1}^1 v_{1l}^2 v_{1(l+m-m_1)}^3, \\
m_{2m}^{(l+1)l} &= \sum_{\substack{l \geq 0 \\ m+l \geq m_2 \geq 0}} v_{1l}^1 v_{1m_2}^2 v_{2(m+l-m_2)}^3 + \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} v_{2m_1}^1 v_{1(l+m-m_1)}^2 v_{1l}^3, \\
m_{1m}^{ll} &= \sum_{l-1 \geq m_2 \geq 0} v_{1(m+l)}^1 v_{1m_2}^2 v_{2(l-m_2-1)}^3 + \sum_{l-1 \geq m_1 \geq 0} v_{2m_1}^1 v_{1(l-m_1-1)}^2 v_{1(m+l)}^3, \\
m_{2m}^{ll} &= \sum_{l-1 \geq m_1 \geq 0} v_{1m_1}^1 v_{2(l-m_1-1)}^2 v_{2(m+l)}^3 + \sum_{l-1 \geq m_2 \geq 0} v_{2(m+l)}^1 v_{2m_2}^2 v_{1(l-m_2-1)}^3, \\
m_{1m}^{l(l+1)} &= \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} v_{1m_1}^1 v_{2(l+m-m_1)}^2 v_{2l}^3 + \sum_{\substack{l \geq 0 \\ l+m \geq m_2 \geq 0}} v_{2l}^1 v_{2m_2}^2 v_{1(l+m-m_2)}^3, \\
m_{2m}^{l(l+1)} &= \sum_{\substack{l \geq 0 \\ l+m \geq m_1 \geq 0}} v_{2m_1}^1 v_{2l}^2 v_{2(l+m-m_1)}^3, \quad m_{1m}^{l(l+2)} = \sum_{l \geq m_1 \geq 0} v_{2m_1}^1 v_{2(m+l+1)}^2 v_{2(l-m_1)}^3, \quad m_{2m}^{l(l+2)} = 0.
\end{aligned}$$

Now, we can estimate $\|\tilde{\mathbf{m}}^{00}\|_{ar}$ as

$$\begin{aligned}
\|\tilde{\mathbf{m}}^{00}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\|_{ar} &= \sum_{j=1,2,m \geq 0} r^{2m+1} \|\tilde{m}_{jm}^{00}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\|_{l_e^a} \leq \sum_{j=1,2,m \geq 0} r^{2m+1} \\
&\times \left(\sum_{\substack{m_2, m_3 \geq 0 \\ m_2+m_3 \leq m-1}} \|v_{1(m-m_2-m_3-1)}^1 v_{2m_2}^2 v_{1m_3}^3\|_{l_e^a} + \sum_{\substack{m_2, m_3 \geq 0 \\ m_2+m_3 \leq m+1}} \|v_{2(m-m_2-m_3-1)}^1 v_{1m_2}^2 v_{2m_3}^3\|_{l_e^a} \right) \\
&\leq \|\mathbf{v}^1\|_{ar} \|\mathbf{v}^2\|_{ar} \|\mathbf{v}^3\|_{ar}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|\mathbf{m}^{(l+2)l}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\|_{ar} &\leq r^{-(4l+4)} \|\mathbf{v}^1\|_{ar} \|\mathbf{v}^2\|_{ar} \|\mathbf{v}^3\|_{ar}, \\
\|\mathbf{m}^{(l+1)l}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\|_{ar} &\leq 3r^{-(4l+2)} \|\mathbf{v}^1\|_{ar} \|\mathbf{v}^2\|_{ar} \|\mathbf{v}^3\|_{ar}, \\
\|\mathbf{m}^{ll}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\|_{ar} &\leq 4r^{-4l} \|\mathbf{v}^1\|_{ar} \|\mathbf{v}^2\|_{ar} \|\mathbf{v}^3\|_{ar}, \\
\|\mathbf{m}^{l(l+1)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\|_{ar} &\leq 3r^{-(4l+2)} \|\mathbf{v}^1\|_{ar} \|\mathbf{v}^2\|_{ar} \|\mathbf{v}^3\|_{ar}, \\
\|\mathbf{m}^{l(l+2)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\|_{ar} &\leq r^{-(4l+4)} \|\mathbf{v}^1\|_{ar} \|\mathbf{v}^2\|_{ar} \|\mathbf{v}^3\|_{ar}.
\end{aligned}$$

Therefore, we see that \mathcal{M} uniformly converges in $\mathcal{L}^3(X_{ar}; X_{ar})$ if $|z| \leq \delta < r$. Thus, we have the conclusion of Lemma 2.4 for $k = 1$. The cases $k \geq 2$ follow from the inductive definition of \mathcal{M}_{2k+1} . \square

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